AS.110.302 (05): Differential Equations and Applications

Recitation Sheets

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Contents

This document records the questions and solutions to the problems reviewed during the recitation for AS.110.302 (05) Differential Equations and Applications in the Spring 2026 semester at the Johns Hopkins University.

- If you notice any error, please contact me via email (sguo45@jhu.edu).

Week 1 (1/20)

Review of Calculus

Course Information.

Topics: First Order ODEs, Higher Order Linear ODEs, Linear System of ODEs, Nonlinear System of ODEs, Series Solution for Second Order ODEs, and Numerical Methods for ODEs.

Schedule: Please attend lectures and recitations using the following times/locations.

	Time	**Location**
Lectures	MWF 1:30pm – 2:20pm	Olin 305
Recitation	T 3:00pm – 3:50pm	Bloomberg 274

Resources:

- Join the office hours of the professor and the TAs (Location and time on Canvas).
- Utilize PILOT learning for the course: https://jhu-ode-pilot.github.io/SP26/.
- If you might have any issues, feel free to contact me via email sguo45@jh.edu.

Assessments Preview: 2 – 3 midterms, quizzes (every other week, from the 3rd week), WebWork (weekly).

Problem I.1. Let k and m be positive integers, evaluate:

$$\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx.$$

Hint: Consider separately when $k = m$ and $k \neq m$.

Solution. First, we consider when $k = m$, we have:

$$\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx = \int_0^\pi 0 dx = 0.$$

Then, consider when $k \neq m$, we have:

$$\begin{aligned}\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx &= \int_0^\pi \big(\sin^2(kx) + \sin^2(mx) - 2\sin(kx)\sin(mx)\big) dx \\ &= \int_0^\pi \left(\frac{1-\cos(2kx)}{2} + \frac{1-\cos(2mx)}{2} - \cos\big((k-m)x\big) + \cos\big((k+m)x\big)\right) dx \\ &= x - \frac{\sin(2kx)}{4k} - \frac{\sin(2mx)}{4m} - \frac{\sin\big((k-m)x\big)}{k-m} + \frac{\sin\big((k+m)x\big)}{k+m}\Bigg|_{x=0}^{x=\pi} = \pi.\end{aligned}$$

Therefore, we have computed that:

$$\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx = \boxed{\begin{cases} 0, & \text{when } k = m, \\ \pi, & \text{when } k \neq m. \end{cases}}$$

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Problem I.2. Evaluate the following indefinite integration:

$$\int x \arcsin x dx.$$

Solution. Here, we first do an integration by parts by picking $u = \arcsin x$ and $dv = xdx$, so we have $du = \frac{dx}{\sqrt{1-x^2}}$ and $v = \frac{x^2}{2}$, so integration by parts gives that:

$$\int x \arcsin x dx = \frac{x^2 \arcsin x}{2} - \int \frac{x^2}{2\sqrt{1-x^2}} dx.$$

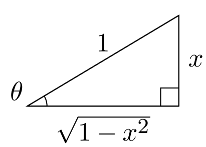
Now, our attention shall focus on the integral:

$$\begin{aligned}\int \frac{x^2}{2\sqrt{1-x^2}} dx &= \int \left(-\frac{1-x^2}{2\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \right) dx \\ &= -\frac{1}{2} \int \sqrt{1-x^2} dx + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \sqrt{1-x^2} dx + \frac{1}{2} \arcsin x + C.\end{aligned}$$

Eventually, we put our attention to the only integration again and do a trigonometric substitution for $x = \sin t$ so $dx = \cos t dt$:

$$\begin{aligned}\int \sqrt{1-x^2} dx &= \int \sqrt{\cos^2 t} \cos t dt = \int \cos^2 t dx = \int \frac{1+\cos 2t}{2} dt \\ &= \frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{\arcsin x}{2} + \frac{\sin(2 \arcsin x)}{4} + C.\end{aligned}$$

Here, we consider visualizing the right triangle with angle the angle θ and hypotenuse as 1.



Therefore, we have $\sin(2 \arcsin x) = 2 \sin(\arcsin x) \cos(\arcsin x) = 2x\sqrt{1-x^2}$. Hence, as we put together all the components, we now have:

$$\begin{aligned}\int x \arcsin x dx &= \frac{x^2 \arcsin x}{2} - \frac{1}{2} \arcsin x + \frac{1}{2} \left(\frac{\arcsin x}{2} + \frac{2x\sqrt{1-x^2}}{4} \right) + C \\ &= \boxed{\frac{x^2 \arcsin x}{2} - \frac{\arcsin x}{4} + \frac{x\sqrt{1-x^2}}{4} + C}.\end{aligned}$$

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Problem I.3. Find the series expansion (centered at 0) and state the radius of convergence:

(a) $f(x) = x \sin x$.

Solution. It is not hard to remember the power series expansion for $\sin x$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

and whose radius of convergence is ∞, so we can easily multiply x over the terms to obtain that:

$$f(x) = x\sin x = x\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+2},$$

and the radius of convergence is still $\boxed{\infty}$. ⌟

(b) $f(x) = \arctan(x)$.

Solution. It is not hard to notice that $f'(x) = \frac{1}{1+x^2}$, whose power series can be written as:

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

with the radius of convergence of 1, so the anti-derivative can be obtained via term-by-term integration:

$$f(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}},$$

with radius of convergence being $\boxed{1}$. ⌟

(c) $f(x) = \int e^{x^2}dx$.

Solution. There is no elementary anti-derivative for e^{x^2}, but we can utilize the term-by-term integration for a good enough approximation. Consider the expansion series for e^x, we have:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

with radius of convergence being ∞, now we then have e^{x^2} as:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!},$$

and the integration could be interpreted as:

$$f(x) = \int e^{x^2}dx = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} + C},$$

with radius of convergence still being $\boxed{\infty}$. ⌟

Week 2 (1/27)

Separable ODEs, Integrating Factor

Reminder: Quiz 1 next week at the start of the recitation section (2/3, 3pm).

Content Review.

Separable ODEs: For ODEs in the form $M(t) + N(y)\frac{dy}{dt} = 0$, it can be separated into:

$$M(t)dt + N(y)dy = 0.$$

Integrating Factor: For ODEs in the form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int_0^t a(s)ds\right),$$

and we multiply it on both sides of the differential equation to obtain $\frac{d}{dt}[\mu(t)y(t)] = b(t)\mu(t)$.

Problem II.1. Find the specific solution to the following initial value problem (IVP):

$$\begin{cases} \dfrac{dy}{dx} = y^2 + 6y + 8, \\ y(0) = 1. \end{cases}$$

Solution. For this problem, we notice that this differential equation is separable, so we can simply turn this into:

$$\frac{dy}{y^2 + 6y + 8} = dx.$$

Now, for the left hand side, we notice that $y^2 + 6y + 8 = (y+2)(y+4)$, so we can do a partial fraction turning it into:

$$\frac{A}{y+2} + \frac{B}{y+4} = \frac{(A+B)y + (4A+2B)}{(y+2)(y+4)},$$

so we have $A + B = 0$ and $4A + 2B = 1$, so we have $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Therefore, we have:

$$\begin{aligned} x + C = \int dx &= \int \left(\frac{1}{2(y+2)} - \frac{1}{2(y+4)}\right) dy \\ &= \frac{1}{2}\ln|y+2| - \frac{1}{2}\ln|y+4| = \frac{1}{2}\ln\left|\frac{y+2}{y+4}\right|, \\ \tilde{C}e^{2x} &= \frac{y+2}{y+4} = 1 - \frac{2}{y+4}, \\ y &= \frac{2}{1 - \tilde{C}e^{2x}} - 4. \end{aligned}$$

When plugging in the initial conditions, we have $1 = \frac{2}{1-\tilde{C}} - 4$, so we have $\tilde{C} = \frac{3}{5}$.
Therefore, the specific solution is:

$$y = \frac{10}{5 - 3e^{2x}} - 4 = \boxed{\frac{12e^{2x} - 10}{5 - 3e^{2x}}}.$$

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Problem II.2. For the following differential equations, find the integrating factor, and solve for the general solutions to the differential equations.

(a) $2y' = y + 1$.

Solution. Note that we should first restore the differential equation back into standard form:

$$\frac{dy}{dt} - \frac{1}{2}y = \frac{1}{2}.$$

Hence, we can compute the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t -\frac{1}{2}ds\right) = \boxed{e^{-\frac{t}{2}}}.$$

By multiplying the integrating factor on both sides of the equation, we have:

$$\frac{d}{dt}(ye^{-\frac{1}{2}t}) = \frac{dy}{dt}e^{-\frac{1}{2}t} - \frac{1}{2}ye^{-\frac{1}{2}t} = \frac{1}{2}e^{-\frac{1}{2}t}.$$

Therefore, we know that the antiderivative is correspondingly:

$$ye^{-\frac{1}{2}t} = \int \frac{1}{2}e^{-\frac{1}{2}t}dt = -e^{-\frac{1}{2}t} + C,$$

and so the solution is $y = \boxed{-1 + Ce^{\frac{1}{2}t}}$. ⌟

(b) $y' + y = \sin 2t$.

Solution. Since we are already in standard form, we can directly compute the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t 1ds\right) = \boxed{e^t}.$$

By multiplying the integrating factors, we obtain that:

$$\frac{d}{dt}(ye^t) = y'e^t + ye^t = e^t \sin 2t.$$

Then, we are left to compute the antiderivative on the right hand side:

$$\begin{aligned} ye^t = \int e^t \sin 2tdt &= e^t \sin 2t - 2\int e^t \cos 2tdt \\ &= e^t \sin 2t - 2\left[\cos 2te^t + 2\int e^t \sin 2tdt\right] \\ &= e^t \sin 2t - 2\cos 2te^t - 4\int e^t \sin 2tdt, \\ 5ye^t = 5\int e^t \sin 2tdt &= e^t \sin 2t - 2e^t \cos 2t + C \\ y &= \boxed{\frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t + Ce^{-t}}. \end{aligned}$$

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Problem II.3. We will be discussing some fundamental logics that helps you to prepare the upcoming theorems on **Existence and Uniqueness** of a differential equation.

In a propositional logic argument, we have an **Antecedent** (P) and a **Consequence** (Q) that can their be true or false. We consider the **Implication** ($P \Rightarrow Q$) being true (T) and false (F) based on the following truth table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) Fill in the following truth table for $Q \Rightarrow P$.

P	Q	$Q \Rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

In mathematics, we use $\neg$ to denote not, that is $\neg P$ means the negation of P, *i.e.*, if P is true, then $\neg P$ is false and vice-versa.

(b) Assume $P \Rightarrow Q$ is true, by the use of truth tables, check if the following conclusions are true, false, or there are insufficient evidence to conclude.

$P \Rightarrow Q$	☑ True.	☐ False.	☐ Insufficient to conclude.
$P \Rightarrow \neg Q$	☐ True.	☐ False.	☑ Insufficient to conclude.
$\neg P \Rightarrow Q$	☐ True.	☐ False.	☑ Insufficient to conclude.
$\neg P \Rightarrow \neg Q$	☐ True.	☐ False.	☑ Insufficient to conclude.
$Q \Rightarrow P$	☐ True.	☐ False.	☑ Insufficient to conclude.
$Q \Rightarrow \neg P$	☐ True.	☐ False.	☑ Insufficient to conclude.
$\neg Q \Rightarrow P$	☐ True.	☐ False.	☑ Insufficient to conclude.
$\neg Q \Rightarrow \neg P$	☑ True.	☐ False.	☐ Insufficient to conclude.

Week 3 (2/3)

Directional Field, Modeling with ODEs

Problem III.1. Let a differential equation be defined as follows:

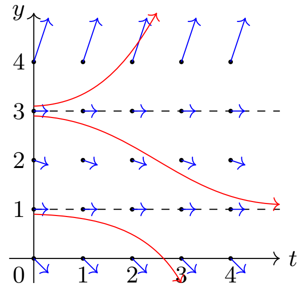
$$\frac{dy}{dt} = y^3 - 5y^2 + 7y - 3.$$

Draw the directional field for the differential equation and sketch a few solutions to this differential equation with some different initial conditions.

Solution. It is not hard to notice that the derivative is independent of t, so we know that:

$$\left.\frac{dy}{dt}\right|_{y=0} = -3, \quad \left.\frac{dy}{dt}\right|_{y=1} = 0, \quad \left.\frac{dy}{dt}\right|_{y=2} = -1, \quad \left.\frac{dy}{dt}\right|_{y=3} = 0, \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{y=4} = 9.$$

Therefore, a sketch of the directional field and some solutions can be shown as follows:



Problem III.2. Solve the following initial value problem for the differential equation:

$$\begin{cases} t\dfrac{dy}{dt} + 2y = \sin t, \\ y(\pi) = \dfrac{2}{\pi}. \end{cases}$$

Solution. Here, we can first make this into standard form, that is:

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{1}{t}\sin t.$$

Then, the integrating factor is:

$$\mu(t) = \exp\left(\int_1^t \frac{2}{s}ds\right) = \exp\left(2\ln|t|\right) = t^2.$$

Therefore, by multiplying the integrating factor as:

$$\begin{aligned} \frac{d}{dt}[t^2y] = t^2\frac{dy}{dt} + 2ty = t\sin t, \\ t^2y = \int t\sin t dt = -t\cos t + \int \cos t dt = -t\cos t + \sin t + C, \\ y = -\frac{1}{t}\cos t + \frac{1}{t^2}\sin t + \frac{C}{t^2}. \end{aligned}$$

Eventually, we plug in the initial condition to obtain that:

$$\frac{2}{\pi} = y(\pi) = \frac{1}{\pi} + \frac{C}{\pi^2} = \frac{\pi + C}{\pi^2},$$

and hence we must have $C = \pi$, so the solution to the differential equation is:

$$y = \boxed{-\frac{1}{t}\cos t + \frac{1}{t^2}\sin t + \frac{\pi}{t^2}}.$$

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Problem III.3. Consider a pool with 100 units of pure water, and 1 unit of sodium hypochlorite (NaOCl) solution in to the pool per hour (and the same amount of well mixed solution in the pool flows out). Assume the input solution contain 10% of NaOCl and 90% of water. Let $a(t)$ model the units of sodium hypochlorite in the pool with respect to time.

(a) Model $a(t)$ with an initial value problem of differential equations.

Solution. Here, we can model the differential equation by the following rule:

$$\frac{da}{dt} = \text{rate of sodium hypochlorite coming in} - \text{rate of sodium hypochlorite flowing out}.$$

We consider the rate of sodium hypochlorite coming in is 0.1 units per hour. The outflow rate is $\frac{a(t)}{100}$, hence, we have the initial value problem of differential equation as:

$$\boxed{\begin{cases} \dfrac{da}{dt} + \dfrac{a}{100} = 0.1, \\ a(0) = 0. \end{cases}}$$

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(b) Solve for the solution of $a(t)$.

Solution. We can find the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t \frac{1}{100} ds\right) = e^{t/100}.$$

Therefore, we have:

$$\begin{aligned} \frac{d}{dt}\left[ae^{t/100}\right] = \frac{da}{dt}e^{t/100} + \frac{1}{100}ae^{t/100} &= 0.1e^{t/100}, \\ ae^{t/100} &= \int 0.1e^{t/100} dt = 10e^{t/100} + C, \\ a(t) &= 10 + Ce^{-t/100}. \end{aligned}$$

By the initial condition, we have $0 = a(0) = 10 + C$, so it implies $C = -10$, so the solution is:

$$a(t) = \boxed{10 - 10e^{-t/100}}.$$

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