AS.110.416: Honors Analysis II

Review

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1 Measure Theory

1.1 Preliminaries

Lemma. Partition of Rectangles.
If a rectangle I is the union of finitely many non-overlapping rectangles, *i.e.*, $I = \sqcup_{k=1}^{\infty} I_k$, then $v(I) = \sum_{k=1}^{N} v(I_k)$.

Lemma. Overlapping Cubes of Rectangles.
If rectangles $I_1, I_2, \cdots, I_N$ satisfy $I \subset \bigcup_{j=1}^{N} I_k$, then $v(I) \leq \sum_{k=1}^{N} v(I_k)$.

Thm. Partition of Open set in $\mathbb{R}$.
Every open set $G \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals.

Thm. Partition of Open set in $\mathbb{R}^n$.
Every open set $G \subset \mathbb{R}^n$ can be written as a countable union of *non-overlapping* (closed) cubes.
Rmk. Dyadic decomposition of $\mathbb{R}^n$ is composed of the cubes has vertex points at $\frac{1}{2^k}\mathbb{Z}$ with length $\frac{1}{2^{k+1}}$.

Prop. Cantor set.
The cantor set C has the following properties:

- $C \neq \emptyset$;
- C has an empty interior, contains no interval, and is totally disconnected;
- C has no isolated points, and all its points are limit points of itself, *i.e.*, C is perfect;
- C is compact;
- $m_*(C) = 0$ (as the union of intervals has length converging to 0).

1.2 Outer Measure

Defn. Outer measure.
Let $E \subset \mathbb{R}^n$, we define the outer/exterior measure of E as:

$$m_*(E) := \inf \sum_{j=1}^{\infty} v(Q_j),$$

where the infimum is taken over all countable covering og E by (closed) cubes, *i.e.*, $E \subset \bigcup_{j=1}^{\infty} Q_j$.

Prop. Properties of Outer measure.
The outer measure of sets follows the below properties:

(i) Closer Approximation: For every $\epsilon > 0$, there exists a covering $E \subset \bigcap_{j=1}^{\infty} Q_j$ with:

$$\sum_{i=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon;$$

(ii) Monotonicity: If $E \subset F$, then $m_*(E) \leq m_* F$;

(iii) Countable Sub-additivity: If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \le \sum_{j=1}^{\infty} m_*(E_j)$;
Rmk. If $m_*(F) = 0$ and $E \subset F$, $m_*(E) = 0$. If $m_*(E_k) = 0$ for all k, then $m_*(\bigcup_{k=1}^{\infty} E_k) = 0$.

(iv) Approximation by Open Sets: Let $E \subset \mathbb{R}^n$, for all $\epsilon > 0$, there exists open set G such that $E \subset G$ and $m_*(G) < m_*(E) + \epsilon$.

(v) Sum of Separated Sets: If $d(E_1, E_2) = \inf\{|x - y| : x \in E_1, y \in E_2\} > 0$, then $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$.
Rmk. This is not true if we only assume $E_1 \cap E_2 = \emptyset$, contradicted by the Banach-Tarski paradox.

(vi) Countable Sum of Almost Disjoints: If a set E is the countable union of almost disjoint cubes, *i.e.*, $E \subset \sqcup_{k=1}^{\infty} Q_k$, then $m_*(E) = \sum_{k=1}^{\infty} v(Q_j)$.

1.3 Measurable sets and Lebesgue measure

Defn. Lebesgue measurable set.
A set $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable if for all $\epsilon > 0$, there exists open set G such that $G \subset E$ and $m_*(G \setminus E) < \epsilon$.
If E is measurable, we define its Lebesgue measure to be $m(E) = m_*(E)$.
Rmk. Countable Sub-additivity ensures that there exists a open set G such that $G \supset E$ and $m_*(G) < m_*(E) + \epsilon$. Then, by Sum of Separated Sets, $G = E \sqcup (G \setminus E)$, then $m_*(G) \le m_*(E) + m_*(G \setminus E)$. If $m_*(E) < \infty$, $m_*(G) - m_*(R) \le m_*(G \setminus E)$.

Prop. Propositions on Measurable Sets.
The following propositions hold for measurable sets:

(i) Every open set is measurable.
Rmk. Every rectangle is measurable.

(ii) Every set with zero outer measure is measurable, which is defined as a *null set*.

(iii) A countably union of measurable sets is also measurable.

(iv) Every closed set is measurable.
Rmk. We first prove that compact sets are measurable and any close sets can be written as a countable union of compact sets, say $F = \bigcup_{k=1}^{\infty}(F \cap B_k)$ where B_k denotes the closed ball of radius k.
Lemma. If F is closed, K is compact, and F, K are disjoint, then $d(F, k) > 0$.
Lemma. If $\{I_k\}_{k=1}^{N}$ is a finite collection of non-overlapping rectangles, then $m\left(\bigcup_{k=1}^{N} I_k\right) = \sum_{k=1}$.

(v) The complement of any measurable set is measurable.
Rmk. Let E be measurable set, there exists H as a countable union of closed sets such that $E^c = H$.

(vi) A countable intersection of measurable sets is measurable.
Cor. If E_1 and E_2 are measurable, $E_1 \setminus E_2$ is measurable, since $E_1 \setminus E_2 = E_1 \cap E_2^c$.

Thm. Countable Additivity.
If $E_1, E_2, \cdots$ are disjoint measurable sets, then $m(\sqcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$.
Lemma. A set E is measurable if and only if for all $\epsilon > 0$, there exists closed set $F \subset E$ such that $m_*(E \setminus F) < \epsilon$.
Cor. Let $\{I_k\}$ be a countable collection of non-overlapping rectangles, then $m\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} m(I_k)$.

Defn. Increasing/Decreasing Subsets of $\mathbb{R}^n$.
If $E_1, E_2, \cdots$ is a countable collection of subsets of $\mathbb{R}^n$ that increases to E in the sense that $E_k \subset E_{k+1}$ for all k, and $E = \bigcup_{k=1}^{\infty} E_k$, then $E_k \nearrow E$.
Similarly, if $E_1, E_2, \cdots$ decreases to E in the sense that $E_{k+1} \subset E$ for all k, and $E = \bigcap_{k=1}^{\infty} E_k$, then $E_k \searrow E$.

Cor. Convergence on Increasing/Decreasing Subsets.
Suppose $\{E_k\}$ is a collection of measurable sets in $\mathbb{R}^n$:

(i) If $E_k \nearrow E$, then $m(E) = \lim_{k\to\infty} m(E_k)$;

(ii) If $E_k \searrow E$ and $m(E_k) < +\infty$ for some k, then $m(E) = \lim_{k\to\infty} m(E_k)$.

Thm. Approximating Sets.
Suppose E is a measurable subset of $\mathbb{R}^n$. Then, for every $\epsilon > 0$:

(i) There exists an open set G with $E \subset G$ and $m(O \setminus E) < \epsilon$;

(ii) There exists a closed set F with $F \subset E$ and $m(E \setminus F) < \epsilon$;

(iii) If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E \setminus K) < \epsilon$;

(iv) If $m(E)$ is finite, there exists a finite union $F = \bigcup_{k=1}^{N} Q_k$ of closed cubes such that $m(E \triangle F) < \epsilon$, where $E \triangle F = (E \setminus F) \cup (F \setminus E)$ is the symmetric difference between E and F.

1.4 σ-Algebra and Borel Sets

Defn. σ-algebra.
A collection Σ of subsets of some universal set U is called a σ-algebra if it satisfies:

(i) $U \in \Sigma$;

(ii) If $E \in \Sigma$, then $E^c \in \Sigma$, where E^c is the complement of E in U;

(iii) If $E_k \in \Sigma$ for all k, then $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

Rmk. The collection of all subsets of $\mathbb{R}^n$ is a σ-algebra.
Rmk. The collection of all Lebesgue measurable sets in $\mathbb{R}^n$ is a σ-algebra, denoted as $\mathcal{M}$.

Defn. Borel σ-algebra.
The smallest σ-algebra containing all open sets in $\mathbb{R}^n$ is called the Borel σ-algebra, denoted as $\mathcal{B}$, or $\mathcal{B}_{\mathbb{R}^n}$.

Elements contained in $\mathcal{B}$ are the Borel sets.

Claim. Intersection being Smallest.
Given a collection Σ_0 of subsets in $\mathbb{R}^n$. Consider the family $\mathcal{F}$ of all σ-algebra that contain Σ_0, *i.e.*, $\mathcal{F} = \{\Sigma : \Sigma \text{ is a } \sigma \text{ algebra and } \Sigma \supset \Sigma_0\}$. Let $\varepsilon := \bigcap_{\Sigma \in \mathcal{F}} \Sigma$. Then:

- ε is a σ-algebra;
- $\varepsilon \supset \Sigma_0$;
- ε is the smallest σ-algebra containing Σ_0, *i.e.*, if ε' is a nother σ-algebra containing Σ_0, then $\varepsilon' \supseteq \varepsilon$.

Rmk. $\mathcal{B} \subsetneq \mathcal{M} \subsetneq \mathcal{P}(\mathbb{R}^n)$, *i.e.*, all Borel sets are measurable.

Defn. G_δ and F_σ Sets: G_σ and F_σ set are the Borel sets, and they are defined as:

(i) The countable intersections of open sets is G_δ sets;

(ii) The countable union of closed sets is F_σ sets.

Thm. Measurable subsets in $\mathbb{R}^n$.
A subset $E \subset \mathbb{R}^n$ is measurable if and only if:

(i) E differs from a G_δ set of measure zero, *i.e.*, $E = H/Z$ where H is a G_δ set and $m(Z) = 0$.

(ii) E differs from a F_σ set of measure zero, *i.e.*, $E = H \cup Z$ where H is a F_σ set and $m(Z) = 0$.

Rmk. $\mathcal{M}$ is a completion of $\mathcal{B}$, *i.e.*, $\mathcal{M}$ is $\mathcal{B}$ adding all null sets.

1.5 Invariance of Lebesgue Measure and Non-Measurable Sets

Prop. Translation-Invariance of Lebesgue Measure.
If $E \in \mathcal{M}_{\mathbb{R}^n}$ and for any $h \in \mathbb{R}^n$, then $E + h := \{x + h | x \in E\}$ is measurable and $m(E + h) = m(E)$.

Prop. Relative Dilation-Invariance of Lebesgue Measure.
If $E \in \mathcal{M}_{\mathbb{R}^n}$ and for any $\delta = (\delta_1, \delta_2, \cdots, \delta_n)$, then $\delta E := \{(\delta_1 x_1, \delta_2 x_2, \cdots, \delta_n x_n) | (x_1, x_2, \cdots, x_n) \in E\}$ is measurable and $m(\delta E) = \delta_1 \cdot \delta_2 \cdots \delta_n m(E)$.
Rmk. Lebesgue measure is reflection-invariant, that is when $E \in \mathcal{M}_{\mathbb{R}^n}$, then $-E := \{-x | x \in E\}$ is measurable and $m(-E) = m(E)$.

Defn. Equivalence Relationship on $[0, 1]$.
An equivalence relation for any $x, y \in [0, 1]$ is defined as follows:

$$x \sim y \text{ if } x - y \in \mathbb{Q}.$$

The equivalence classes are $[x] := \{x + q \in [0, 1] : q \in \mathbb{Q}\}$. The equivalence classes either are disjoint or coincide, and they form a partition of $[0, 1] = \bigsqcup_{\alpha \in A} x_\alpha$.

Axiom. The Axiom of Choice.
Consider a family of non-empty, pairwise disjoint sets $\{E_\alpha\}_{\alpha \in A}$ in a common set X, there exists a subset

of X which contains exactly one element from each E_α for $\alpha \in A$.
In other words, there exists a function $\alpha \mapsto x_\alpha$ (known as a "choice" function) such that $x_\alpha \in E_\alpha$ for all α.

Defn. Vitali Set.
Let V be a set consisting of exactly one element from each disjoint equivalent class $[x_\alpha]$ of $[0,1]$.

Thm. The Vitali Set is not measurable.
Rmk. This is by the translated set $v_k = v + q_k = \{x + q_k : x \in V\}$ where $\{q_k\}$ is an enumeration of rationals in $[-1,1] \cap \mathbb{Q}$. The inclusion $[0,1] \subset \bigsqcup_{k=1}^{\infty} v_k \subset [-1,2]$, thus $1 \leq \infty \times m(v) \leq 3$, which is a contradiction.

1.6 Measurable Functions

Defn. Measurability of a Function.
Consider real-valued function f defined on a measurable set $E \subset \mathbb{R}^n$ such that $f : E \to \mathbb{R} \cup \{\pm\infty\}$. f is measurable if for any $a \in \mathbb{R}$, $\{x \in E : f(x) < a\}$ (denoted as $\{f < a\}$) is measurable.
Rmk. f is finite-valued if $-\infty < f(x) < +\infty$ for all $x \in E$.

Cor. Equivalent Definitions of Measurable Function.
f is measurable if and only if $\{f \leq a\}$, or $\{f > a\}$, or $\{f \geq a\}$ is measurable for all $a \in \mathbb{R}$.
If f is finite valued, then f is measurable if and only if $\{a < f < b\}$ is measurable for all $a, b \in \mathbb{R}$.

Defn. Almost Everywhere.
A property if said to hold almost everywhere in E if it holds in E except for a subset of E with measure zero.

Prop. Propositions on Measurable Functions.
The following properties on measurable functions holds:

(i) A finite-valued function f is measurable if and only if $f^{-1}(G)$ is measurable for every open set $G \subset \mathbb{R}$.

(ii) If f is continuous on $\mathbb{R}^n$, then f is measurable.
Rmk. If f is measurable and finite-valued, and Φ is continuous on $\mathbb{R}$, then $\phi \circ f$ is measurable.

(iii) Suppose $\{f_k\}_{n=1}^{\infty}$ is a sequence of measurable function on E. Then:
$$\sup_n f_n(x), \qquad \inf_n f_n(x), \qquad \limsup_{n\to\infty} f_n(x), \qquad \text{and} \qquad \liminf_{n\to\infty} f_n(x)$$
are measurable.
Rmk. Note that we can have $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$, and $\inf_n f_n(x) = -\sup_n(-f_n(x))$.
Rmk. The upper and lower limits can be written as $\limsup_{n\to\infty} f_n(x) = \inf_k \{\sup_{n\geq k} f_n\}$ and $\liminf_{n\to\infty} f_n(x) = \sup_k \{\inf_{n\geq k} f_n\}$.

(iv) If $\{f_k\}_{k=1}^{\infty}$ is a collection of measurable function, and $f(x) = \lim_{k\to\infty} f_k(x)$, then f is measurable.

(v) If f and g are measurable, then:

- The integer powers of f^k for $k \geq 1$ are measurable;
Rmk. For odd powers, $\{f^k > a\} = \{f > a^{1/k}\}$ and for even power, $\{f^k > a\} = \{f > a^{1/k}\} \cup \{-f < a^{1/k}\}$.

- $f+g$ and $f \cdot g$ is measurable if both f and g are finite-valued.
  Rmk. In this case, we note that $\{f+g>a\} = \{f>a-g\} = \bigcup_{q\in\mathbb{Q}}\{f>q>a-g\}$ and $fg = \frac{1}{4}\left[(f+g)^2-(f-g)^2\right]$.

(vi) Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is measurable.

1.7 Approximation Measurable Functions by Simple Functions

Defn. Characteristic Functions.
The characteristic function (or indicator function) of a set E is defined as:

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Defn. Step Functions.
A step function is a finite function of the form:

$$f(x) = \sum_{k=1}^{N} a_k \chi_{R_k}(x),$$

where $a_1, a_2, \cdots, a_N \in \mathbb{R}$ and $R_1, R_2, \cdots, R_N$ are rectangles.

Defn. Simple Functions.
A simple function is a finite function of the form:

$$f(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x),$$

where $a_1, a_2, \cdots, a_N \in \mathbb{R}$ and $E_1, E_2, \cdots, E_N$ are measurable sets of finite measure.
Rmk. We can assume without the loss of generality that E_k's are disjoint and a_k's are distinct.

Thm. Approximating Non-Negative Measurable Functions by Simple Functions.
Suppose f is a non-negative measurable function. There exists an increasing sequence of non-negative simple functions $\{\varphi_k(x)\}_{k=1}^{\infty}$ that converges to f, *i.e.*:

$$\varphi_k(x) \le \varphi_{k+1}(x) \text{ and } \lim_{k\to\infty} \varphi_k(x) = f(x) \text{ for all } x.$$

Rmk. Here, we define $\varphi_k(x)$ as:

$$\varphi_k(x) = \begin{cases} k, & \text{if } f(x) \ge k \text{ and } |x| < k, \\ \dfrac{j-1}{2^k}, & \text{if } f(x) \in \left[\dfrac{j-1}{2^k}, \dfrac{j}{2^k}\right], j \in \{1, 2, \cdots, k\cdot 2^k\}, \\ 0, & \text{if } |x| \ge k. \end{cases}$$

Thm. Approximating Measurable Functions by Simple Functions.
Suppose f is a measurable function. There exists a sequence of simple function $\{f_k\}_{k=1}^{\infty}$ that satisfies:

$$|\varphi_k(x)| \le |\varphi_{k+1}(x)| \text{ and } \lim_{k\to\infty} \varphi_k(x) = f(x) \text{ for all } x.$$

Rmk. In particular, we have $|\varphi_k(x)| \leq |f(x)|$ for all x and k.
Rmk. The proof is made possible with the construction that:

$$\begin{cases} f^+ := \max\{f, 0\}, \\ f^- := -\min\{f, 0\}, \end{cases}$$

so that $f^\pm$ are non-negative measurable functions, where they are respectively approximated by $\left\{\varphi_k^{(1)}(x)\right\}_{k=1}^\infty$ and $\left\{\varphi_k^{(2)}(x)\right\}_{k=1}^\infty$, respectively. Therefore, we have $\varphi_k(x) = \varphi_k^{(1)} - \varphi_k^{(2)}$.

Thm. Approximating Measurable Functions by Step Functions.
Suppose f is measurable on $\mathbb{R}^n$, then there exists a sequence of step functions $\{\psi_k\}_{k=1}^\infty$ that converges pointwise to $f(x)$ for almost every x.
Rmk. This case can be thought of as an extended case for approximating by simple functions. For every $\epsilon > 0$, we can always find $Q_1, Q_2, \cdots, Q_N$ such that $m(E \triangle \bigcup_{j=1}^N Q_j) \leq \epsilon$ for all E. By considering the grid formed by extending the sides of these cubes, we see that there exist almost disjoint rectangles, and there are smaller rectangles R_j contained in those rectangles forming a collection of disjoint rectangles such that $m\left(E \triangle \bigsqcup_{j=1}^M R_j\right) \leq 2\epsilon$. Thus, we have:

$$\psi(x) = \sum_{j=1}^M \chi_{R_j}(x).$$

Rmk. For each approximation, it is converging except possibly a set of measure $\leq 2\epsilon$. However, all the variations set $E_k := \{x : f(x) \neq \psi(c)\}$ in which $m(E_k) \leq 2\epsilon$ and by having $F_K = \bigcup_{j=K+1}^\infty E_j$ and $F = \bigcap_{K=1}^\infty F_K$, we have $m(F) = 0$ and $\psi_k(x) \to f(x)$ for all x in the complement of F.

1.8 Littlewood's 3 Principles of Real Analysis

Intuition. Littlewood's 3 Principles of Real Analysis: Littlewood summarized the connections in the form of three principles that provide a useful intuitive guide in the initial study of the theory:

(i) Every measurable set is nearly a finite union of cubes;

(ii) Every measurable function is nearly continuous;

(iii) Every almost everywhere convergent sequence of functions is nearly uniformly converged.

Rmk. "Nearly" means that the set of exceptions has small measure.

Thm. Measurable Set Nearly as a Finite Union of Cubes:
(Approximating Sets (iv):) If $m(E)$ is finite, there exists a finite union $F = \bigcup_{k=1}^N Q_k$ of closed cubes such that $m(E \triangle F) < \epsilon$, where $E \triangle F = (E \setminus F) \cup (F \setminus E)$ is the symmetric difference between E and F.

Thm. Egorov's Theorem.
Suppose $\{f_k\}_{k=1}^\infty$ is a sequence of measurable function that converges almost everywhere to a finite-valued function f on a measurable set E of finite measure. Then, for all $\eta > 0$, there exists a closed set $F \subset E$ such that:

$$m(E \setminus F) < \eta \text{ and } f_k \rightrightarrows f \text{ on } F.$$

Lemma. Under the same assumption, for all $\epsilon > 0$ and $\eta > 0$, there exists closed set $F \subset E$ and $N \in \mathbb{N}$ such that:

$$m(E \setminus F) < \eta \text{ and } |f(x) - f_k(x)| < \epsilon \text{ for all } x \in F \text{ and } k \geq N.$$

Rmk. For $E = \mathbb{R}^1$ and $f_k(x) = \chi_{[-k,k]}(x)$ converges pointwise to $f(x) \equiv 1$ since the measure is not finite.

Thm. Lusin's Theorem.
Suppose f is measurable and finite-valued measurable function on a measurable set E. Then for all $\epsilon > 0$, there exists closed set $F \subset E$ such that $m(E \setminus F) < \epsilon$ and $f|_F$ is continuous.
Lemma. A simple measurable function f on a measurable set E satisfies the condition that for all $\epsilon > 0$, there exists closed set $F \subset E$ such that $m(E \setminus F) < \epsilon$ and $f|_F$ is continuous.

2 Integration Theory

2.1 Lebesgue Integral for Simple Functions

Defn. Canonical Form of Simple Function.
The canonical form of a simple function is:

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}(x),$$

where a_j's are distinct and non-zero and E_k's are disjoint and measurable sets with finite measure.

Defn. Lebesgue Integral on Simple Functions.
The Lebesgue Integral for $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$ is:

$$\int \varphi(x)dx := \sum_{j=1}^{N} a_j m(E_j).$$

Rmk. The integration of φ is the same for any representation.

Prop. Properties on Lebesgue Integral for Simple Function.
The following properties holds for Lebesgue integration for simple function:

(i) Linearity: $\int (a\varphi + b\varphi) = a \int \varphi + b \int \varphi$;

(ii) Additivity: Let E be a measurable set with finite measure, then we have $\int_E \varphi = \int \varphi \cdot \chi_E$;
Rmk. If E and F are disjoint subsets of $\mathbb{R}^n$ with finite measure, then $\int_{E \sqcup F} \varphi = \int_E \varphi + \int_F \varphi$.

(iii) Monotonicity: Let $\varphi \leq \psi$, them $\int \varphi \leq \int \psi$;
Rmk. In particular, if $\varphi = \psi$ almost everywhere, then $\int \varphi = \int \psi$.

(iv) Triangular Inequality: If φ is a simple function, so is $|\varphi|$, and $|\int \varphi| \leq \int |\varphi|$.

2.2 Lebesgue Integral for Bounded Function Supported on a Set of Finite Measure

Defn. Support of Function.
The support of a function f is defined as:

$$\text{supp}(f) := \{f \neq 0\}.$$

f is supported on a set E if $f = 0$ outside of E, *i.e.*, $\text{supp}(f) \subset E$.
In this stage, we are interested in f being bounded, measurable such that $m(\text{supp}(f)) < +\infty$.
For such functions, there exists a sequence of simple functions $\{\varphi_n\}_{n=1}^{\infty}$ with each φ_n bounded and supported on a finite measurable set, and $\varphi_n(x) \to f$ for all x.

Thm. Convergence of Simple Approximation Function.
Let f be a bounded function supported on a set E of finite measure. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M, supported on E, and with $\varphi_n(x) \to f(x)$ or a.e. x, then:

(i) The limit $\lim_{n\to\infty} \varphi_n(x)dx$ exists;
Rmk. Here, we have that $-M\chi_E \leq \varphi_k \leq M\chi_E$.
Rmk. The proof wants to show that $\{\int \varphi_k\}_{k=1}^{\infty}$ is a Cauchy sequence.

(ii) If $f = 0$ a.e., then the limit $\lim_{n\to\infty} \int \varphi_n = 0$.

Defn. Lebesgue Integral on Bounded Function Supported on a Set of Finite Measure.
For a bounded function f supported on a set of finite measure, the integral is:

$$\int f(x)dx = \lim_{n\to\infty} \int \varphi_n(x)dx,$$

where $\{\varphi_n(x)\}_{n=1}^{\infty}$ is any sequence of simple functions satisfying that:

- $|\varphi_N| < M$;
- Each φ_n is supported on a support of f;
- $\varphi_n(x) \to f(x)$ for a.e. x as n tends to $+\infty$.

Rmk. We need to show that the definition is independent with the choice of sequence. Suppose $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ are two qualified sequences, then we have $\{\eta_n\}_{n=1}^{\infty}$ with $\eta_n = \varphi_n - \psi_n$, in which $\{\eta_n\}_{n=1}^{\infty}$ is consisted of simple functions bounded by $2M$, supported on a set of finite measure, and $\eta_n \to 0$ a.e. as n tends to $+\infty$. Hence, the two limits $\lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int \psi_n$.

Prop. Properties on Lebesgue Integral for Bounded Function Supported on a Set of Finite Measure.
The properties remains the same as for bounded function supported in a set of finite measure:

(i) Linearity: $\int(af + bg) = a\int f + b\int g$;

(ii) Additivity: If E and F are disjoint subsets of $\mathbb{R}^n$ with finite measure, then $\int_{E\sqcup F} f = \int_E f + \int_F f$;

(iii) Monotonicity: Let $f \leq g$, them $\int f \leq \int g$;
Rmk. In particular, if $f = g$ almost everywhere, then $\int f = \int g$;

(iv) Triangular Inequality: $|f|$ is also bounded, and $|\int f| \leq \int |f|$.

Thm. Bounded Convergence Theorem.
Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions bounded by M and supported on a set E of finite measure, in which $f_k \to f$ a.e. as $k \to \infty$. Then, f is measurable, bounded, and supported on E for a.e. Moreover:

$$\int |f_n - f| \to 0 \text{ as } n \to \infty,$$

hence implying that:

$$\int f_n \to \int f \text{ as } n \to \infty.$$

Rmk. In constructing this theorem, by Egorov's Theorem, there exists closed sets $F_\eta \subset E$ such that $f_n \rightrightarrows f$ on F_η, and by $m(E \setminus F_\eta)$ implies that $\int |f_n - f| = \int_{F_\eta} |f_n - f| + \int_{E \setminus F_\eta} \leq \epsilon m(E) + 2M\eta$.

Thm. Riemann and Lebesgue Integral.
Suppose $f(x)$ is Riemann integrable on $[a, b]$. Then f is Lebesgue measurable, and:

$$\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx.$$

Rmk. The Riemann integral is based on bounded functions, and it uses a partition by Γ which forms two sequences of step function, which is:

$$\{\varphi_k\}_{k=1}^{\infty} \text{ and } \{\psi_k\}_{k=1}^{\infty},$$

in which each element is absolutely bounded by M and:

$$\varphi_1(x) \leq \varphi_2(x) \leq \cdots \leq f(x) \leq \cdots \leq \psi_2(x) \leq \psi_1(x).$$

By definition of Riemann integral, we have that:

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \varphi_k(x)dx = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \psi_k(x)dx = \int_{[a,b]}^{\mathcal{R}} f(x)dx.$$

By the definition of the step functions, the integrals on $\varphi_k(x)$ and $\psi_k(x)$ are equal for Riemann and Lebesgue integration. Let $\widetilde{\varphi}$ and $\widetilde{\psi}$ be their respective limits, then $\widetilde{\varphi} \leq f \leq \widetilde{\psi}$. As they are both measurable, then the bounded convergence theorem, the integrals converges at the limit, which gives:

$$\int_{[a,b]}^{\mathcal{L}} \left(\widetilde{\varphi}(x) - \widetilde{\psi}(x)\right) dx = 0,$$

which then implies $\widetilde{\varphi} = \widetilde{\psi}$ a.e., thus f is measurable. Then by $\varphi_k \to f$ a.e., we have the two integrations generating the same result.

2.3 Lebesgue Integral for Non-negative Measurable Function

Defn. Lebesgue Integral for Non-negative Measurable Function.
Let $f \geq 0$ be a measurable function, we defined:

$$\int f(x)dx := \sup_g \int g(x)dx,$$

where the supremum is tajkn over all measurable functions g such that $0 \leq g \leq f$ and g is bounded and supported on a set of finite measure.
Def. f is Lebesgue measurable if $\int f(x)dx < +\infty$.

Prop. Properties on Lebesgue Integral for Non-negative Measurable Function.
The following properties holds:

(i) Linearity: For $a, b > 0$, $\int(af + bg) = a\int f + b\int g$;

(ii) Additivity: If E and F are disjoint subsets of $\mathbb{R}^n$ with finite measure, then $\int_{E \sqcup F} f = \int_E f + \int_F f$.

(iii) Monotonicity: Let $0 \leq f \leq g$, them $\int f \leq \int g$;
Rmk. Note that $\int g$ can be $+\infty$ as we are not assuming that g is integrable;

(iv) If g is integrable, and $0 \leq f \leq g$, then f is integrable;

(v) If f is integrable, then $f < +\infty$ a.e.;

(vi) If $\int f = 0$, then $f = 0$ a.e.

Lemma. Fatou's Lemma.
Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of non-negative measurable functions such that $f_k \to f$ a.e. Then:

$$\int f \leq \liminf_{n\to\infty} \int f_k.$$

Rmk. By construction, $\int f = \sup_{0 \leq g \leq f, \text{bounded and supported}} \int g$, if we let $g_k := \min\{g, f_k\} \leq g$, thus it is bounded and supported by $\text{supp}(g)$. By the bounded convergence theorem, we have $\int g = \lim_{n\to\infty} \int g_k \leq \int f_k$ and since $\int g_k \leq \int f_k$, we have that:

$$\int f = \lim_{k\to\infty} \int g_k \leq \liminf_{n\to\infty} \int f_k.$$

Cor. Monotone Convergence Theorem.
Suppose f is a non-negative measurable function, and $\{f_k\}_{k=1}^{\infty}$ is a sequence of non-negative measurable function with $f_n(x) \leq f(x)$ and $f_k(x) \to f(x)$ for a.e. x. Then $\lim_{k\to\infty} \int f_k = \int f$.
Cor. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of non-negative measurable functions such that $f_k \nearrow f$, then $\lim_{k\to\infty} \int f_k = \int f$.
Rmk. By Fatou's Lemma, $\int f \leq \liminf_{k\to\infty} \int f_k$ and $f_k \leq f$ implies that $\int f_k \leq \int f$ and hence $\limsup_{k\to\infty} \int f_k \leq \int f$.

Cor. Monotone Convergence Theorem for Series.
Consider the series $\sum_{k=1}^{\infty} a_k(x)$, where $a_k(x) \geq 0$ is measurable for every $k \geq 1$. Then:

$$\int \left(\sum_{k=1}^{\infty} a_k(x) \right) dx = \sum_{k=1}^{\infty} \left(\int a_k(x) dx \right).$$

Rmk. If $\sum_{k=1}^{\infty} \left(\int a_k(x) dx \right)$ is finite, then $\sum_{k=1}^{\infty} a_k(x) dx$ converges for a.e. x.
Rmk. This is $f_j(x) = \sum_{k=1}^{j} a_k(x) \nearrow \sum_{k=1}^{\infty} a_k(x)$ through monotone convergence theorem.

2.4 Lebesgue Integral for Measurable Function

Defn. Lebesgue Integral for Measurable Function:
Let f be measurable function. f is integrable if $|f|$ is integrable (as $|f| = f^+ + f^-$).

Hence, the Lebesgue Integral of f is defined to be:

$$\int f := \int f^+ - \int f^-.$$

Prop. Properties of Lebesgue Integrable functions.
The properties remains the same as for general integrable functions:

(i) Linearity: $\int (af + bg) = a \int f + b \int g$;

(ii) Additivity: If E and F are disjoint subsets of $\mathbb{R}^n$ with finite measure, then $\int_{E \sqcup F} f = \int_E f + \int_F f$;

(iii) Monotonicity: Let $f \leq g$, them $\int f \leq \int g$;

(iv) Triangular Inequality: $|f|$ is also bounded, and $|\int f| \leq \int |f|$.

Prop. Integral Converging to Zero for Some Set.
Suppose f is integrable on $\mathbb{R}^n$. Then for every $\epsilon > 0$:

(i) There exists a ball B such that $\int_{B^c} |f| < \epsilon$;
Rmk. The integrable functions does not necessarily vanishes near ∞, that is if f is integrable, then $\lim_{|x| \to \infty} f(x) = 0$ is false.
Rmk. We may consider B_k as ball centered at origin with radius k, in which $f_k := f \cdot \chi_{B_k} \nearrow f$. Hence by monotone convergence theorem, we have $\lim_{k \to \infty} \int f_k = \int f < \infty$ and thus $|\int f - \int f_k| = \left| \int_{B_k^c} f \right| < \epsilon$ for $k \geq N$.

(ii) There exists $\delta > 0$ such that $\int_E |f| < \epsilon$ for any measurable set E such that $m(E) < \delta$.

Thm. Dominance Convergence Theorem.
Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable function such that $f_k \to f$ a.e. Assume that $|f_k| \leq g$ a.e. where g is integrable. Then $\lim_{k \to \infty} \int f_k = \int f$.
Rmk. In fact, $\int |f_k - f| \to 0$ as $k \to +\infty$.
Rmk. Let $-g \leq f_k \leq g$, then we can have $\int (f + g) \leq \liminf_{k \to \infty} \int (f_k + g)$ by Fatou's Lemma. Then, likewise, we have $-\int f \leq \liminf_{k \to \infty} (-\int f_k) = -\limsup_{n \to \infty} \int f_k$.

Defn. Complex-valued Functions: A complex-valued function can be written as:

$$f(x) = u(x) + iv(x), \text{ where } u(x) = \mathfrak{Re} f(x) \text{ and } \mathfrak{Im} f(x).$$

Rmk. Hence, f is integrable if $|f| := \sqrt{|u|^2 + |v|^2}$ is integrable, that is if and only if u and v are integrable.

Defn. Lebesgue Integral over Complex-valued Functions.
The Lebesgue integral of complex valued is defined to be:

$$\int f(x)dx = \int u(x)dx + i \int v(x)dx.$$

Rmk. Addition and scalar multiplication is closed for complex-valued f measurable function on E.
Rmk. The collection of all complex-valued integrable functions on a measurable subset $E \subset \mathbb{R}^n$ forms a vector space over $\mathbb{C}$.

2.5 The Space of Integrable Functions

Def. Norm in Space of Integrable Functions $L^1(E)$.
For any $f \in L^1(\mathbb{R}^n)$, we define the norm of f to be:

$$\|f\|_{L^1} := \int_{\mathbb{R}^n} |f(x)| dx,$$

where the norm induces the following properties:

(i) Linearity: $\|\lambda f\|_{L^1} = |\lambda| \cdot \|f\|_{L^1}$ for all $\lambda \in \mathbb{C}$;

(ii) Triangle Inequality: $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$;

(iii) $\|f\|_{L^1} = 0$ implies that $f = 0$ a.e. on $\mathbb{R}^n$;

(iv) $d(f,g) := \|f-g\|_{L^1}$ induces $L^1(\mathbb{R}^n)$ into a metric space.

Thm. $L^1(\mathbb{R}^n)$ is Complete.
$L^1(\mathbb{R}^n)$ is complete with the metric $d(f,g) = \|f-g\|_{L^1}$.
Cor. If f is convergent to $f \in L^1$, then there is a subsequence $\{f_{k_j}\}_{k_j \in \mathbb{Z}^+}$ of $\{f_n\}_{n=1}^{\infty}$ so that $f_{k_j} \to f$ pointwise a.e. x.
Rmk. This is not necessarily true if we want the entire sequence to converge to f.

Defn. Dense Families of Function.
A family of integrable function G is dense in $L^1(\mathbb{R}^n)$ if for all $f \in L^1(\mathbb{R}^n)$ and for all $\epsilon > 0$, there exists $g \in G$ such that $\|f-f\|_{L^1} < \epsilon$.

Lemma. Dense Families in $L^1(\mathbb{R}^n)$.
The following families are dense in $L^1(\mathbb{R}^n)$:

(i) Simple functions;

(ii) Step functions;

(iii) Continuous functions with compact support, denoted $C_C(\mathbb{R}^n)$.

Strategy. Strategy in Proving Properties for $L^1(\mathbb{R}^n)$.
If we want to prove some properties for all integrable functions, we:

(i) prove the property holds for a dense family;

(ii) Use a limiting argument to conclude for all $L^1(\mathbb{R}^n)$.

Appl. Invariance of Lebesgue Integral.
The following invariance holds for Lebesgue integration with $f \in L^1(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, and $\delta > 0$:

$$\begin{cases} \int_{\mathbb{R}^n}^{\mathcal{L}} f(x-h)dx = \int_{\mathbb{R}^n}^{\mathcal{L}} f(x)dx; \\ \delta^n \int_{\mathbb{R}^n}^{\mathcal{L}} f(\delta x)dx = \int_{\mathbb{R}^n}^{\mathcal{L}} f(x)dx; \\ \int_{\mathbb{R}^n}^{\mathcal{L}} f(-x)dx = \int_{\mathbb{R}^n}^{\mathcal{L}} f(x)dx. \end{cases}$$

Rmk. The proof was made first on simple functions. Then, for the complex-valued functions, the conclusions can be made from $f_h = \chi_{E_h}$, which holds for all $L^1(\mathbb{R}^n)$.
Cor. By such, we can conclude the commutativity for convolution of f and g by:

$$f * g(x) := \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy = g * f(x).$$

Appl. Translation and Continuity.
For any $f \in L^1(\mathbb{R}^n)$, then $\|f_h - f\| \to 0$ as $h \to 0$, where $f_h = f(x+h)$.
Rmk. The proof follows along the continuous function with compact support, say $g \in C_C(\mathbb{R}^n)$ in which $|g(x-h) - g(x)| < \epsilon$ for all $x \in \mathbb{R}^n$ if $|h| < \delta$, in which the argument follows quickly through:

$$\begin{aligned}\|f_h - f\|_{L^1} &= \int |f_h - f| \\ &= \int |f_h - g_h + g_h - g + g - f| \leq \int |f_h - g_h| + \int |g_h - g| + \int |g - f| \\ &= 2\|f-g\|_{L^1} + \|g_h - g\|_{L^1} < 3 \times \frac{\epsilon}{3} < \epsilon\end{aligned}$$

as $|h| < \delta$.

2.6 Fubini's Theorem

Defn. Slices and Mapped Functions.
Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and function $f(x,y)$ be defined on $E := \mathbb{R}^m \times \mathbb{R}^n$, the slices are defined as:

$$\begin{aligned}E_x &:= \{y \in \mathbb{R}^n : (x,y) \in E\}, \\ E^y &:= \{x \in \mathbb{R}^m : (x,y) \in E\}.\end{aligned}$$

At the same time, we concern the following functions:

$$\begin{aligned}f_x(y) &:= f(x,y), \\ f^y(x) &:= f(x,y).\end{aligned}$$

Thm. Fubini's Theorem.
Let $f \in L^1(\mathbb{R}^{m+n})$, then:

(i) for a.e. $x \in \mathbb{R}^m$, the slice f_x is measurable and integrable in $\mathbb{R}^n$,

(ii) the function $x \mapsto \int_{\mathbb{R}^n} f(x,y)dy$ is defined for a.e. $x \in \mathbb{R}^m$, measurable and integrable on $\mathbb{R}^m$, and

(iii) $\iint_{\mathbb{R}^{m+n}} f(x,y)dxdy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)dy \right) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)dx \right) dy.$

Rmk. The proving strategy is to let the family of functions satisfying Fubini's Theorem as $\mathcal{F}$, and prove by following steps:

(i) prove that $\mathcal{F}$ is closed under linear combination, so we reduce the proof to non-negative functions,

(ii) prove that $\mathcal{F}$ contains the limit of monotonic sequences, then we reduce the proof to simple, thus characteristic functions,

(iii) prove that for E being a G_δ-set in $\mathbb{R}^{m+n}$ with finite measure, then $\chi_E \in \mathcal{F}$,

(iv) prove that for N being a null set in $\mathbb{R}^{m+n}$, then $\chi_N \in \mathcal{F}$, and the slices N_x are also null set in $\mathbb{R}^n$, by such, we know that this applies for all finite measurable set,

(v) for any $f \in L^1(\mathbb{R}^{m+n})$, then $f \in \mathcal{F}$.

Rmk. The converse is not necessarily true. If f is measurable in $\mathbb{R}^{m+n}$, and $T := \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)dy\right) dx$ is finite, f is not necessarily integrable.

Thm. Tonelli's Theorem.
Let $f(x,y)$ be non-negative measurable function in $\mathbb{R}^{m+n}$, then:

(i) for a.e. $x \in \mathbb{R}^n$, the slice f_x is measurable in $\mathbb{R}^n$,

(ii) the function $x \mapsto \int_{\mathbb{R}^n} f_x dy$ (taking values in $\mathbb{R}^+ \cup \{+\infty\}$) is measurable, and

(iii) $\iint_{\mathbb{R}^{m+n}} f(x,y)dxdy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)dy\right) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)dx\right) dy$. (This could be infinite).

Rmk. Fubini-Tonelli Theorem.
We use the two theorems in the following cases:

(i) Use Tonelli's theorem on $|f|$ to show that $f \in L^1(\mathbb{R}^{m+n})$, and then

(ii) use Fubini for $\iint_{\mathbb{R}^{m+n}} f(x,y)dxdy$.

Rmk. In proving Tonelli's Theorem, we construct that:

$$f_k(x,y) := \begin{cases} 0, & \text{if } |(x,y)| > k, \\ \min\{f(x,y), k\}, & \text{if } |(x,y)| \leq k. \end{cases}$$

Lemma. Exterior Measure on Product of Sets.
Let $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$, then:

$$m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2),$$

so if one set has exterior measure zero, then the exterior measure of product must be zero.

Prop. Measure of Product of (Measurable) Sets.
Let $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$ be measurable, then $E := E_1 \times E_2$ is measurable in $\mathbb{R}^{m+n}$, and:

$$m(E) = m(E_1)m(E_2),$$

so if one set has measure zero, then the measure of product must be zero.

Cor. Suppose f is a non-negative function on $\mathbb{R}^n$, and let:

$$\mathcal{A} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Then:

(i) f is measurable on $\mathbb{R}^d$ if and only if $\mathcal{A}$ is measurable on $\mathbb{R}^{n+1}$,

(ii) if the conditions in (i) holds, then $\int_{\mathbb{R}^n} f(x)dx = m_{\mathbb{R}^{n+1}}(\mathcal{A})$.

3 Differentiation

3.1 Differentiation of the Integral

Defn. Average of Integration.
Let $f \in L^1(\mathbb{R}^n)$, consider the set function $\mathcal{M}(\mathbb{R}^n) \ni E \mapsto \int_E f$, and we let:

$$⨍_E f = \frac{1}{m(E)} \int_E f.$$

Thm. Lebesgue Differentiation Theorem.
Let $f \in L^1(\mathbb{R}^n)$, then:

$$\lim_{Q \to x} \frac{1}{m(Q)} \int_Q f = f(x),$$

for a.e. $x \in \mathbb{R}^n$.
Rmk. Q works for cubes and balls, but only certain classes of rectangles works.

3.2 Hardy-Littlewood Maximal Function

Def. Hardy-Littlewood Maximal Function.
Let $h \in L^1(\mathbb{R}^n)$, we define its Hardy-Littlewood maximal function of h as:

$$\mathcal{M}h(x) = h^*(x) := \sup_{Q \ni x} \frac{1}{m(Q)} \int_Q |h|.$$

Rmk. The Hardy-Littlewood maximal function of $f \in L(\mathbb{R}^n)$ follows:

- $0 \le f^*(x) \le +\infty$,
- For any $\lambda > 0$, $\{f^* > \lambda\}$ is open in $\mathbb{R}^n$ implies that f^* is measurable,
- f^* might not be in $L^1(\mathbb{R}^n)$.

Thm. Hardy Littlewood Theorem.
If $f \in L^1(\mathbb{R}^n)$, then f^* belongs to weak $L^1(\mathbb{R}^n)$, namely, there exists a constant C (independent of f and α) such that $\forall \alpha > 0$:

$$m(\{f^* > \alpha\}) \le \frac{C}{\alpha} \int_{\mathbb{R}^n} |f|.$$

Lemma. Elementary Version of Vitali Lemma.
Suppose $\mathcal{F} = \{Q_1, \cdots, Q_N\}$ is a finite collection of (open or closed) cubes in $\mathbb{R}^n$. Then $\exists$ a disjoint sub-collection $Q_{i_1}, Q_{i_2}, \cdots, Q_{i_e}$ of $\mathcal{F}$ such that:

$$m\left(\bigcup_{i=1}^N Q_i\right) \le 3^n \sum_{j=1}^{\ell} m(Q_{i_j}),$$

i.e.:

$$3^{-n} m\left(\bigcup_{i=1}^N Q_i\right) \le m\left(\bigsqcup_{j=1}^{\ell} Q_{i_j}\right).$$

Defn. Locally Integrable.
f is locally integrable ($f \in L^1_{\text{loc}}(\mathbb{R}^n)$) if $f \in L^1(B)$ for any ball B in $\mathbb{R}^n$. Lebesgue Differentiation Theorem holds if we assume $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.
Rmk. For any measurable set $E \subset \mathbb{R}^n$, $\chi_E \in L^1_{\text{loc}}(\mathbb{R}^n)$, but not necessarily in Ł$^1(\mathbb{R}^n)$.

Defn. Lebesgue Density Point.
Let E be a measurable set and $x \in \mathbb{R}^d$, x is a point of Lebesgue density of E if:

$$\lim_{m(B)\to 0, x\in B} \frac{m(B\cap E)}{m(B)} = 1.$$

Rmk. A.e. $x \in E$ is a Lebesgue density point of E and a.e. $x \notin E$ is not a Lebesgue density point of E.

Defn. Lebesgue Point.
A point x is referred as a Lebesgue point of f if:

$$\lim_{Q\to x} ⨍_Q |f(y) - f(x)| dy = 0,$$

and this holds for a.e. $x \in \mathbb{R}^n$.

Cor. Almost Every Point is Lebesgue.
If $f \in L_{\text{loc}}(\mathbb{R}^n)$, then a.e. $x \in \mathbb{R}^n$ is Lebesgue point.

3.3 Approximation to Identity

Defn. The Scaling Function.
Let k be a bounded integrable function such that $\int k = 1$ in $\mathbb{R}^n$. Then the scaling function is:

$$k_\delta(x) := \frac{1}{\delta^n} k\left(\frac{x}{\delta}\right).$$

The scaling is due to the fact that:

$$\int_{\mathbb{R}^n} k_\delta(x)dx = \int_{\mathbb{R}^n} \frac{1}{\delta^n} k\left(\frac{x}{\delta}\right) dx = \int_{\mathbb{R}^n} k(x)dx = 1.$$

Rmk. By the same token, we have $\int_{\mathbb{R}^n} |k_\delta| = \int_{\mathbb{R}^n} |k|$.
Rmk. If k has compact support, say B_{R_0}, then k_δ is supported on $B_{\delta R_0}$.

Defn. Good Kernels.
A good kernel $K_\delta(x)$ is integrable and satisfies the following for all $\delta > 0$:

(i) $\int_{\mathbb{R}^d} K_\delta(x)dx = 1$,

(ii) $\int_{\mathbb{R}^d} |K_\delta(x)|dx \le A$, and

(iii) for every $\eta > 0$, $\int_{|x|\ge\eta} |K_\delta(x)|dx \to 0$ as $\delta \to 0$,

where A is a constant depending on δ.
Prop. Properties with $f * k_\delta$.
For any integrable function f in $\mathbb{R}^n$, consider the convolution $(f * k_\delta)(x)$, which is integrable that:

- Let k be a bounded integrable function in $\mathbb{R}^n$, such that $\int k = 1$, and suppose k has compact support, then:
$$(f * k_\delta)(x) \to f(x) \text{ as } \delta \to 0,$$
for any x that is a Lebesgue point of f.
- Let k be a bounded integrable function in $\mathbb{R}^n$ such that $\int k = 1$. Then $f * k_\delta \to f$ in L^1 as $\delta \to 0^+$.
- Let k be a bounded integrable function in $\mathbb{R}^n$ such that $\int k = 1$. Suppose $k(x) = \mathcal{O}\left(\frac{1}{|x|^{n+\lambda}}\right)$ for some $\lambda > 0$ (*i.e.*, $|k(x)| \leq \frac{c}{|x|^{n+\lambda}}$ for $|x|$ large enough). Then $f * k_\delta(x) \to f(x)$ for x which is a Lebesgue point of f.
- If $k \in C_c^m(\mathbb{R}^n)$, then $f * k$ is continuous and bounded.

Rmk. By (ii), the convergence in L^1 implies that there exists $\delta_k \to 0^+$ such that $f * k_{\delta_j}(x) \to f(x)$ for a.e. x.
Rmk. For (iii), we have that:
$$\frac{1}{|x|^n}\chi_{\{|x|>1\}} \notin L^1(\mathbb{R}^n), \ \frac{1}{|x|^{n+\epsilon}}\chi_{\{|x|>1\}} \in L^1(\mathbb{R}^n).$$

Rmk. For (iv), we have that:
$$\partial_{x_i}(f * k(x)) = f * (\partial_{x_i} K(x)).$$

Ex. Kernels for PDEs:

- The Poisson kernel is:
$$P_y(x) := \frac{1}{y} K\left(\frac{x}{y}\right) = \frac{1}{\pi}\frac{y}{x^2 + y^2},$$
for the upper half plane Laplace equation.
- The heat kernel is:
$$H_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)},$$
solving the global Cauchy for Heat equation.

Lemma. Average Function.
Suppose that f is integrable on $\mathbb{R}^d$, and that x is a Lebesgue point of f. Let:
$$\alpha(r) = \frac{1}{r^n}\int_{|y|\leq r} |f(x-y) - f(x)| dy, \text{ whenever } r > 0.$$
Then $\alpha(r)$ is continuous function of $r > 0$, and $\alpha(r) \to 0$ as $r \to 0$ and $\alpha(r)$ is bounded for all $r > 0$.

4 Hilbert Space

4.1 $L^2(\mathbb{R}^n)$ Space

Defn. L^2 Space.
$L^2(\mathbb{R}^n)$ is the collection of complex-valued measurable functions in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f(x)|^2 dx < +\infty$.
The L^2-norm of f is defined as $\|f\|_{L^2} := \left(\int |f(x)|^2 dx\right)^{1/2}$.
Rmk. The following holds:

(i) For $\lambda \in \mathbb{C}$, $\|\lambda f\|_{L^2} = |\lambda| \cdot \|f\|_{L^2}$.

(ii) For $f, g \in L^2(\mathbb{R}^n)$, and if $f = g$ a.e., then $\|f - g\|_{L^2} = 0$ (identified as the same element).

(iii) $f \in L^2(E)$ if $f \cdot \chi_E \in L^2(\mathbb{R}^n)$.

(iv) For $1 \le p < +\infty$, $\|f\|_{L^p} = (\int |f(x)|^p dx)^{1/p}$.

Defn. Inner Product in L^2.
On $L^2(\mathbb{R}^n)$, we define the inner product as:

$$\langle f, g \rangle = \int f(x) \cdot \overline{g(x)} dx.$$

Rmk. We check that $f\overline{g}$ is integrable as $\int |f\overline{g}| = \int |f| \cdot |g| \le \int \frac{1}{2}(|f|^2 + |g|^2) < +\infty$. (if $a, b > 0$, then $ab \le \frac{1}{2}(a^2 + b^2)$).
Rmk. Cauchy-Schwartz Inequality indicates $|\langle f, g \rangle| \le \|f\|_{L^2} \cdot \|g\|_{L^2}$.

Prop. Properties on the L^2 Space.

(i) Inner product $\langle \bullet, \bullet \rangle$ satisfies Cauchy-Schwartz.

(ii) For any $g \in L^2(\mathbb{R}^n)$ fixed, $f \in L^2(\mathbb{R}^n) \mapsto \langle f, g \rangle \in \mathbb{C}$ is linear in f and $\langle g, f \rangle = \overline{\langle f, g \rangle}$.

(iii) $L^2(\mathbb{R}^n)$ is a vector space over C and $\| \bullet \|_{L^2}$ is a norm. (Distance is $d(f, g) = \|f - g\|$.)

Thm. L^2 Space is Complete.
The space of $L^2(\mathbb{R}^n)$ is complete with respect to the metric from the norm, *i.e.*, all Cauchy sequences converges.
Rmk. The proof involves the construction of:

$$S_K(f)(x) = f_{n_1}(x) + \sum_{k=1}^{K} (f_{n_{k+1}}(x) - f_{n_k}(x)), \text{ and } S_K(g)(x) = |f_{n_1}(x)| + \sum_{k=1}^{K} |f_{n_{k+1}}(x) - f_{n_k}(x)|,$$

where f_{n_k} is subsequence in which the L^2 norm of there differences are within 2^{-k}. Then, $\|S_K(g)\|$ with MCT implies that $f \in L^2$ and the construction of $S_K(f)$ supports that f_{n_k} converges to f by DCT. Eventually, by triangle inequality:

$$\|f_n - f\| \le \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon.$$

Thm. L^2 Space is Separable.
The space $L^2(\mathbb{R}^n)$ is separable, in the sense that there exists a countable collection $\{f_k\}$ of elements in $L^2(\mathbb{R}^d)$ such that their linear combinations are dense in $L^2(\mathbb{R}^d)$.
Rmk. Here, we constructed the collection $\mathcal{C}$ of characteristic functions χ_D, where D is a dyadic cube in $\mathbb{R}^n$, with coefficients being complex numbers whose real and imaginary parts are rational, *i.e.*, $D := \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$ for integers j and k.

4.2 Hilbert Space

Defn. Hilbert Space.
A set $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$ if:

(H1) $\mathcal{H}$ is a vector space over $\mathbb{C}$.

(H2) $\mathcal{H}$ is equipped with an inner product $\langle \bullet, \bullet \rangle$ such that:

- For any $g \in \mathcal{H}$ fixed, $f \mapsto \langle f, g \rangle$ is linear on $\mathcal{H}$.
- $\langle f, g \rangle = \overline{\langle g, f \rangle}$.
- $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}$ with equality if and only if $f = 0$ in $\mathcal{H}$.

(P) Properties: $\|f\| = \langle f, f \rangle^{1/2}$ and Cauchy-Schwartz with Triangle Inequality holds.

(H3) $\mathcal{H}$ is complete with respect to the metric $d(f, g) = \|f - g\|$. (not required for Pre-Hilbert Space, but Pre-Hilbert Space can be extended to Hilbert Space, called the completion of the Pre-Hilbert Space by having objects as all Cauchy sequences).

(H4) $\mathcal{H}$ is separable, *i.e.*, $\mathcal{H}$ has a dense subset which is countable.

Rmk. Banach space is a normed vector space with (H3).

Ex. Examples of Hilbert Space.

(i) $\left(L^2(\mathbb{R}^n), \langle \bullet, \bullet \rangle\right)$ is a Hilbert space over $\mathbb{C}$.

(ii) $\mathbb{C}^N := \left\{(z_1, \cdots, z_N) : z_i \in \mathbb{C}\right\}$ with for $z, w \in \mathbb{C}^N$ that $\langle z, w \rangle = \sum_{i=1}^N z_i \overline{w_i}$ (or the standard Euclidean inner product) is a Hilbert space.

(iii) $\ell^2(\mathbb{Z}) := \left\{(\cdots, a_{-1}, a_0, a_1, \cdots) : a_i \in \mathbb{C}, \sum_{-\infty}^{\infty} |a_n|^2 < \infty\right\}$ with inner product being the infinite sum of the product $a_k \overline{b_k}$ is a Hilbert Space (also classified as (i)).

(iv) $W^{1,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n)\}$ with $\langle f, g \rangle = \langle f, g \rangle_{L^2} + \sum_{i=1}^n \langle \partial_i f, \partial_i g \rangle$ is a Hilbert space (also classified as (i)).

Rmk. All the Hilbert space can be classified as (i) or (ii).

4.3 Orthogonality and Basis

Defn. Orthogonality.
$f, g \in \mathcal{H}$ are orthogonal, *i.e.* $f \perp g$ if $\langle f, g \rangle = 0$.
Rmk. Pythagorean theorem: If $f \perp g$, then $\|f + g\|^2 = \|f\|^2 + \|g\|^2$.

Defn. Orthonormal Collection.
A collection $\{e_\alpha\}_{\alpha \in A}$ in $\mathcal{H}$ is orthonormal if $\langle e_\alpha, e_\beta \rangle = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$.
Rmk. Since $\mathcal{H}$ has a countable dense subset, any orthonormal collection in $\mathcal{H}$ has at most countably many element (since the separation has to be $\|e_\alpha - e_\beta\| = \|e_\alpha\|^2 + \|-e_\beta\|^2 = 2$).

Prop. Projection onto Orthonormal Collection.
If $\{e_k\}$ is orthonormal in $\mathcal{H}$, and $f = \sum_{k=1}^N a_k e_k \in \mathcal{H}$, then $f\|^2 = \sum_{k=1}^N |\langle f, e_k \rangle|^2$.

Defn. Orthonormal Basis.
An orthonormal collection $\{e_k\}$ of $\mathcal{H}$ is an orthonormal basis if the finite linear combination of e_k's over $\mathbb{C}$ are dense in $\mathcal{H}$.

Thm. Equivalent Conditions for Orthonormal Collection.
Let $\{e_k\}$ be an orthonormal collection in $\mathcal{H}$, the following are equivalent:

(i) Finite linear combinations of $\{e_k\}$ are dense in $\mathcal{H}$.

(ii) If $f \in \mathcal{H}$ and $\langle f, e_j \rangle = 0$ for all $j \in \mathbb{N}$, then $f = 0$.

(iii) If $f \in \mathcal{H}$ and $S_N(f) = \sum_{k=1}^{N} a_k e_k \in \mathcal{H}$ with $a_k := \langle f, e_k \rangle$, then $S_N(f) \to f$ in the norm as $N \to +\infty$. (Namely, $\sum_{k=1}^{N} \langle f, e_k \rangle e_k \to f$.)

(iv) (Parseval's Identity) If $f \in \mathcal{H}$, then $\|f\|^2 = \sum_{k \in \mathbb{N}} |\langle f, e_k \rangle|^2$.

Rmk. All above vases implies that the basis is orthonormal.

Thm. Orthonormal Basis of Hilbert Space.
Every Hilbert space has an orthonormal basis.
Rmk. The construction is by Gram-Schmidt process.

4.4 Unitary Mapping

Defn. Unitary Isomorphisms.
Given 2 Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, with $(\langle \bullet, \bullet \rangle_{\mathcal{H}}, \langle \bullet, \bullet \rangle_{\mathcal{H}'})$, a mapping $T : \mathcal{H} \to \mathcal{H}'$ is a unitary isomorphism if:

(i) T is a linear map, *i.e.*, $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in \mathcal{H}$.

(ii) T is a bijection.

(iii) $\|T(f)\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$.

Rmk. (iii) guarantees that inner product is preserved, *i.e.*:

$$\langle f, g \rangle = \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2 + i\left(\left\|\frac{F}{i} + G\right\|^2 - \left\|\frac{F}{i} - G\right\|^2\right)\right].$$

Cor. Unitary Isomorphisms for Infinite Dimensional Hilbert Spaces.
Any two infinite dimensional Hilbert spaces are unitarily equivalent, *i.e.*, there exists a unitary isomorphism between them.
Rmk. The construction is by enumerating an orthonormal basis $\{e_1, e_2, \cdots\}$ and $\{e_1', e_2', \cdots\}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and have $T : \mathcal{H}_1 \to \mathcal{H}_2, e_i \mapsto e_i'$.

4.5 Fourier Series

Appl. Conventions to $L^2([-\pi, \pi])$ Space.
We consider $L^2([-\pi, \pi])$ with inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$.

Prop. Orthonormal Basis in $L^2([-\pi, \pi])$.
$\{e^{-ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2([-\pi, \pi])$.
Rmk. By Euler's Formula, we can construct another orthonormal basis of $\{\cos kx, \sin kx\}_{k\in\mathbb{N}}$.
Rmk. If f is piecewise continuous (or Riemann integrable) on $[-\pi, \pi]$, then $f \in L^2([-\pi, \pi])$, which extend f to be defined on $\mathbb{R}$ with periodicity of 2π.

Thm. Approaching from Fourier Series.
We write the Fourier series of $f(x)$ (integrable on $[-\pi, \pi]$) as:

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

then:

(i) If $a_k = 0$ for all $k \in \mathbb{Z}$, then $f(x) = 0$ a.e. x.

(ii) $\sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ikx} \to f(x)$ for a.e. x as $r \to 1^-$.

Rmk. (ii) is a consequence of the Poisson kernel.

Thm. Convergence of Fourier Series.
Suppose $f \in L^2([-\pi, \pi])$, then:

(i) (Parseval's Relation) $\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$.

(ii) The mapping $f \mapsto \{a_n\}$ is a unitary correspondence between $L^2([-\pi, \pi])$ and $\ell^2(\mathbb{Z})$.

(iii) The Fourier series of f converges to f in the L^2-norm, that is:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 dx \to 0 \text{ as } N \to \infty,$$

where $S_N(f) = \sum_{n=-N}^{N} a_n e^{inx}$.

Rmk. If $f \in L^2([-\pi, \pi])$ and $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$, then $f'(x) = \sum_{k=-\infty}^{\infty} k a_k e^{ikx}$, thence:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{k=-\infty}^{\infty} |k a_k|^2.$$

Therefore, $f'(x) \in L^2([-\pi, \pi])$ is a better decay for $|a_k|$ as $k \to \pm\infty$.

5 Abstract Measure Space

5.1 Abstract Measure

Defn. Measure Space.
A measure space on a set X is a triple $(X, \mathcal{M}, \mu)$ where:

(i) $\mathcal{M}$ is a σ-algebra, which is a non-empty collection of subsets of X closed under complements, countable unions, and countable intersections. Elements in $\mathcal{M}$ are the measurable sets.

(ii) $\mu : \mathcal{M} \to [0, +\infty]$ is a function satisfying that for any countable collection of disjoint sets in $\mathcal{M}$, $E_1, E_2, \cdots$ satisfies $\mu\left(\bigsqcup_k E_k\right) = \sum_k \mu(E_k)$. $\mu(E)$ is the measure of E.

Rmk. (Lebesgue-Radon-Nikodym Theorem) All the measures must be a combination of the following:

(i) Let $X = \{x_k\}$, $\mathcal{M} = \mathcal{P}(X)$, define $\mu(\{x_k\}) = \mu_k$ where $\{m_k\}$ is a sequence of numbers in $[0, +\infty]$. For any $E \in \mathcal{M}$, we have $\mu(E) = \sum_{k:x_k \in E}^{\mu_k}$.

(ii) Let $X \in \mathbb{R}^n$, $\mathcal{M} = \{\text{Lebesgue measurable sets}\}$ and for any $E \in \mathcal{M}$, $\mu(E) = \int_E f dx$ where f is a given non-negative measurable function on $\mathbb{R}^n$.

5.2 Exterior Measure

Defn. Outer Measure.
An outer measure on a set X is a function μ_* from all subsets of X to $[0, +\infty]$ satisfying that:

(i) $\mu_*(\emptyset) = 0$.

(ii) If $E_1 \subset E_2$, then $\mu_*(E_1) \leq \mu_*(E_2)$.

(iii) For any countable collection of sets $E_1, E_2, \cdots$ in X, $\mu_*(\bigcup_k E_k) \leq \sum_k \mu_*(E_k)$.

Defn. Carathéodory Measurable Sets.
Given $E \subset X$, E is Carathéodory measurable if for any $A \subset X$:

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c).$$

Rmk. This is equivalent to the definition of Lebesgue measurable sets.
Rmk. By (iii) in outer measure, $\mu_*(A) \leq \mu_*(A \cap E) + \mu_*(A \cap E^c)$ is satisfies.

Thm. Outer Measure Forms Measure.
Given a outer measure μ_* on a set X, the collection $\mathcal{M}$ of all Carathéodary measurable set form a σ-algebra. Moreover, μ_* restricted to $\mathcal{M}$ is a measure.
Rmk. Any set of outer measure 0 is Carathéodory measurable. Since if $\mu_*(Z) = 0$, then $\mu_*(A) \geq \mu_a st(A \cap Z) + \mu_*(A \cap Z^c) = \mu_a st(A \cap Z^c)$ by monotonicity.
Defn. σ-finite.
We say a measure space is $(X, \mathcal{M}, \mu)$ is σ-finite if X can be written as the union of countably many measurable sets of finite measure.

Defn. Borel Algebra.
The Borel σ-algebra, $\mathcal{B}_x$ denotes the smallest σ-algebra containing all open sets.

Defn. Metric Outer Measure.
An outer measure μ_* on (X, d) is a metric outer measure if:

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B) \text{ for any } A, B \subset X,$$

such that:

$$d(A,B) := \inf\{d(x,y) : x \in A, y \in B\} > 0.$$

Thm. Metric Outer Measure Forms Measure.
If μ_* is a metric outer measure on (X,d), then Borel sets in X are Carathéodory measurable and μ_* restricted to $\mathcal{B}_x$ is a measure.
Rmk. From the previous theorem, $\mathcal{M}$ is a σ-algebra already. Then, we need to show that all open/closed sets are Carathéodory measurable. Here for a closed set F, we define $E_k := \{x \in A \cap F^c : d(x,F) \geq \frac{1}{k}\}$. We prove that $\lim_{k\to\infty} \mu_*(A \cap F^c)$ by letting $C_k := E_{k+1} \setminus E_k$.

Defn. Borel Set.
Given a metric space (X,d), a measure μ defined on all Borel sets of X is the Borel Set.

Prop. Suppose the Borel measure μ is finite on all balls in X with finite radii, then for any Borel set E, any $\epsilon > 0$, there exists open set $G \supset E$, closed set $F \subset E$ such that $\mu(G \setminus E) < \epsilon$ and $\mu(E \setminus F) < \epsilon$.

Lemma. Convergence for Monotone Sequences.
Let $(X, \mathcal{M}, \mu)$ be a measure space, if measurable sets $E_k \nearrow E$, then $\mu(E_k) \nearrow \mu(E)$.

5.3 Pre-Measure

Defn. Pre-Measure.
Given a set X, an algebra in X is a non-empty collection of subsets of X that are closed under complements, finite unions, and finite intersections. A pre-measure on an algebra $\mathcal{A}$ is a function $\mu_0 : \mathcal{A} \to [0, +\infty]$ that satisfies:

- $\mu_0(\emptyset) = 0$.
- If $A_1, A_2, \cdots$ is a countable collection of disjoint sets in $\mathcal{A}$ with $\bigsqcup_j A_j \in \mathcal{A}$, then:
$$\mu_0\left(\bigsqcup_k A_k\right) = \sum_k \mu_0(A_k).$$

Lemma. The Extension Theorem.
If μ_0 is a pre-measure on an algebra $\mathcal{A}$, define an outer measure μ_* on any subset E of X as:

$$\mu_*(E) = \inf\left\{\sum_j \mu_0(A_j) : E \subset \bigcup_j A_j \text{ where } A_j \in \mathcal{A} \text{ for all } j\right\}.$$

Then μ_* is an outer measure on X that satisfies:

(i) $\mu_*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

(ii) Any set in $\mathcal{A}$ is Carathéodary measurable with respect to μ_*.

Rmk. The extension is unique. Let $\mathcal{M}$ be a σ-algebra containing $\mathcal{A}$, let μ be the measure generated from μ_*. Assume that μ is σ-finite, then for any other measure ν defined on $\mathcal{M}$ such that $\nu = \mu$ on sets in $\mathcal{A}$,

$\nu(E) = \mu(E)$ for any $E \in \mathcal{M}$.

Appl. Product Measure.
Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be 2 σ-finite measure space. We construct a measure space on $X := X_1 \times X_2$ by having the measure:

$$\mu_0(A \times B) = \mu_1(A) \cdot \mu_2(B).$$

Here, we have that $\mathcal{A}$ as the smallest algebra containing all measurable rectangles. Note that for all products as the disjoint union of rectangles, we have:

$$\mu_0(A \times B) = \sum \mu_0(A_j \times B_j).$$