`EN.553.361` : Introduction to Optimization

Notebook

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Contents

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- The document contains lecture contents, notes, and adapts contents from the following text:
  - Bazaraa, Jarvis, and Sherali, Linear Programming and Network Flows, Wiley.
- The document might contain minor typos or errors. Please point out any notable error(s).

Part 1

Introduction

I Preliminaries on Optimization

I.1 Optimization Problem

There are options and there are costs associated to the option. The goal of **optimization** is to find the option that maximizes the rewards and minimize the costs.

Different choices have different outcomes, and we would want use algorithms and develop tools to find the best choice. Here is an example of optimization:

Example I.1.1. You have up to 6 units of two different nutrients can be added to the soil, and we require a number of units of nutrients 2 that is at least the (natural) logarithm of the number of units of nutrient 1. We denote x_1 as the number of units of nutrient 1 and x_2 as the number of units of nutrient 2, and our goal is to maximize the expected height of the plant, which is modeled by:

$$H(x_1, x_2) := 1 + x_1^2(x_2 - 1)^3 e^{-x_1 - x_2}.$$

Here, we would consider the feasible region as follows:

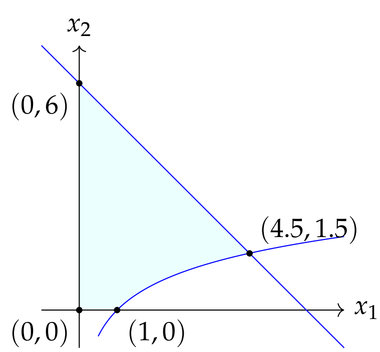


Figure I.1. Feasible region of the above problem with critical points.

We want to maximize based on:

$$\begin{aligned} \max\ & 1 + x_1^2(x_2 - 1)^3 e^{-x_1 - x_2}, \\ \text{s.t. } & x_1 + x_2 \leq 6, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned} \tag{$\hat{P}$}$$

There will be 4 vertices, namely $(0, 6)$, $(4.5, 1.5)$, $(0, 0)$, and $(1, 0)$.
In fact, for this problem, the maximum possible expected height can be attained when $x_1 = 4.5$ and $x_2 = 1.5$. ◊

With the previous example, we can naturally develop the model of the generic type of optimization problem.

Definition I.1.2. Generic Optimization Problem.
Let $S \subset \mathbb{R}^n$ with some function $f : S \to \mathbb{R}$. The **optimization problem** (denoted P) would be:

$$\begin{aligned} &\text{min (or max) } f(x), \\ &\text{s.t. } x \in S. \end{aligned}$$

Here, $f(x)$ is the **objective function**, x is the **decision variables**, and S is the **feasible region**. In general, we have $x \in S$ as the **constraints**. ⌟

Remark I.1.3. Note when the codomain of f is $\mathbb{R}$, the maximization problem can be turned into a minimization problem since we can simply *negate* the problem. ⌟

Definition I.1.4. Global Minimizers.
x^* is a (global) **minimizer** if it satisfies:

(i) x^* is feasible, *i.e.*, $x^* \in S$, and

(ii) For all $y \in S$, $f(x^*) \leq f(y)$.

x^* is a strict (global) **minimizer** if it satisfies:

(i) x^* is feasible, *i.e.*, $x^* \in S$, and

(ii) For all $y \in S \setminus \{x^*\}$, $f(x^*) < f(y)$. ⌟

However, the minimizer is not guaranteed to exist, as some of the following examples portraits:

Example I.1.5. First, we construction the problem as follows:

$$\begin{aligned} &\min \log x, \\ &\text{s.t. } 0 < x \leq 7. \end{aligned}$$

It is trivial that the objective function value is unbounded below, and there is not a possible minimum.
Likewise, if we turn the problem into:

$$\begin{aligned} &\min \log x, \\ &\text{s.t. } 1 < x \leq 7. \end{aligned}$$

Here, the objective function value is bounded below, but there is still not solution.
Even easier, we can simply construct a infeasible problem:

$$\begin{aligned} &\min \log x, \\ &\text{s.t. } x \leq 0.5, x > 1, \end{aligned}$$

since there are no viable choices of x, since $\{x : x \leq 0.5\} \cap \{x : x > 1\} = \emptyset$.
Well, we can of course have a problem with a minimizer solution, say:

$$\begin{aligned} &\min \log 3 + (x-2)^2, \\ &\text{s.t. } 1 \leq x \leq 3, \end{aligned}$$

by trivial calculus, we have $x = 2$ as the minimizer (c.f. **second derivative test**).
Similarly, if we consider:

$$\begin{aligned} &\min \log 3 + (x-2)^2, \\ &\text{s.t. } x \geq 10. \end{aligned}$$

This problem resembles a (strictly) monotonic objective function, and the constraints work here. ◊

In fact, we can anticipate a "compact" (closed and bounded in $\mathbb{R}^n$) condition for a solution on a continuously defined objective function (c.f. **extreme value theorem**).

I.2 Mathematical Preliminaries

Definition I.2.1. Euclidean Length and Distance.
For any $x \in \mathbb{R}^n$, the **Euclidean length** of x is:

$$\|x\| := \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

For any $x, y \in \mathbb{R}^n$, the **Euclidean distance** between x and y is $\|x - y\|$. ⌟

Specifically, the vector $x - y$ can be thought of an arrow pointing into the space:

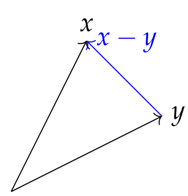


Figure I.2. Vector subtraction in $\mathbb{R}^n$.

Definition I.2.2. Neighborhood.
For any $x \in \mathbb{R}^n$ and $\epsilon > 0$, the ϵ-neighborhood is:

$$N_\epsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \epsilon\}.$$

⌟

Example I.2.3.

- Consider $\mathbb{R}^1$ with $7 \in \mathbb{R}$, we have $N_3(7) = (4, 10)$:

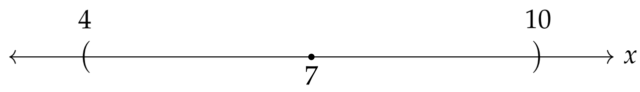
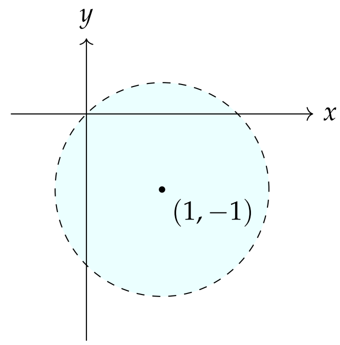


Figure I.3. $N_3(7)$ on the number line.

Note that 4 and 10 are not included in the neighborhood.

- Consider $\mathbb{R}^2$ with $(1, -1) \in \mathbb{R}^2$, we can visualize it as:



- For $\mathbb{R}^3$, $N_1\big((0,0,0)\big)$ is the interior of a ball centered at the origin with radius of 1.

◊

Definition I.2.4. Interior Point and Boundary Point.
For any set $S \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ is:

- an **interior point** of S if there exists a neighborhood of x which is a subset of S.
- a **boundary point** of S if for all neighborhoods contain a point in S and a point in S^c.

⌟

Lemma I.2.5. Give any set S and some $x \in S$, x is either the interior of S, the interior of S^c, or the boundary point of S.

Here, we can note that a point would have to be one of the three above cases, and only one of the three above cases.

Definition I.2.6. Open and Closed Sets.
Let $S \subset \mathbb{R}^n$, it is:

- an **open set** if each point in S is an interior point of S.
- a **closed set** if S contains all of its boundary points.

⌟

Note that a set can be neither open nor closed, and a set can be open and closed.

Example I.2.7. $(1, 2] \subset \mathbb{R}$ is not open and not closed, and $\emptyset \in \mathbb{R}$ is open and closed. ◊

Definition I.2.8. Bounded Set.
$S \subset \mathbb{R}^n$ is **bounded** if S is a subset of a neighborhood of a point in $\mathbb{R}^n$. ⌟

Theorem I.2.9. Suppose a set $S \subset \mathbb{R}^n$ is closed and bounded, and suppose $f : S \to \mathbb{R}$ is continuous, then $\min f(x)$ such that $x \in S$ exists.

In *Real Analysis*, this theorem will be formatted as the extreme value theorem which is casted on **compact sets**, which happens to be equivalent to closed and bounded in finite dimensional Euclidean space (c.f. **Heine-Borel theorem**).

Proposition I.2.10. For any $S \subset \mathbb{R}^n$, S is open if and only if S^c is closed.

Consider the following problem (or program):

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } x \in S, \end{aligned} \tag{P}$$

and we suppose $x^* \in S$. x^* is a global minimizer precisely when for all $y \in S$, then $f(x^*) \leq f(y)$ from Definition I.1.4. However, being a global minimizer condition is rather strong, and hence, we would want to introduce a weaker condition as a local minimizer:

Definition I.2.11. Local Minimizers.
Suppose $x^* \in S$:

- x^* is a **local minimizer** precisely when there exists neighborhood of x^* $N \subset \mathbb{R}^n$ such that for all $y \in S \cap N$, $f(x^*) \leq f(y)$.
- x^* is a strict **local minimizer** precisely when there exists neighborhood of x^* $N \subset \mathbb{R}^n$ such that for all $y \in S \cap N \setminus \{x^*\}$, $f(x^*) < f(y)$.

⌟

Note that this is distinct with the "global minimizers" that is in Definition I.1.4, and it is trivial that a (strict) **global minimizer** is a (strict) **local minimizer**.

Part 2

Linear Optimization

II Linear Programming

II.1 Introduction to Linear Programming

Example II.1.1. Diet Problem.
You will pick levels of four ingredients to go into chicken feed. First, we denote the following:

$$x_1 := \text{\# units of ingredient 1},$$
$$x_2 := \text{\# units of ingredient 2},$$
$$x_3 := \text{\# units of ingredient 3},$$
$$x_4 := \text{\# units of ingredient 4}.$$

Now, given minimum requirements of 3 nutrients as follows:

$$n_1 \geq 6.2, \qquad n_2 \geq 11.9, \qquad \text{and} \qquad n_3 \geq 10.$$

Also given how many units of each nutrient are in each of the ingredients:

nutrient \ ingredient	1	2	3	4
1	1.2	2.6	0	9.2
2	3.9	1	0.8	2
3	6	0	4	3.1

Eventually, we give the cost per unit of each ingredient:

$$c_1 = 6.2, \qquad c_2 = 2, \qquad c_3 = 1.6, \qquad \text{and} \qquad c_4 = 3.2.$$

The goal is to find the blend of ingredients satisfying nutritional requirements at minimum cost.
We can turn the problem into the following optimization problem:

$$\begin{aligned} \min\ & 6.2x_1 + 2x_2 + 1.6x_3 + 3.2x_4, \\ \text{s.t. } & 1.2x_1 + 2.6x_2 + 0x_3 + 9.2x_4 \geq 6.2, \\ & 3.9x_1 + 1x_2 + 0.8x_3 + 2x_4 \geq 11.9, \\ & 6x_1 + 0x_2 + 4x_3 + 3.1x_4 \geq 10, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

$\diamondsuit$

Definition II.1.2. A **linear program** in canonical forms would be portrayed as:

$$\begin{aligned} \min\ & c_1x_1 + c_2x_2 + \cdots + c_nx_n, \\ \text{s.t. } & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \geq b_1, \\ & a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \geq b_2, \\ & \vdots \\ & a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \geq b_m, \\ & x_1, x_2, \cdots, x_n \geq 0, \end{aligned}$$

in which we can transformed into as follows:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t. } & Ax \geq b, \\ & x \geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, and the inequality here is *coordinate*-wise. ⌟

Remark II.1.3. The **linear programming** satisfies the following conditions:

- **Proportionality**: Each variable contributes proportionally to the objective functions and constraint functions,
- **Additivity**: The contribution of each variables adds up into the objective functions and constraint functions,
- **Divisibility**: We do not require integer input but rather real numbers.

⌟

Example II.1.4. Transportation Problem.
There are three electricity generation plants α, β, and γ, and there are two cities u and v.
Suppose α, β, and γ produces 63.2, 98.6, and 32.5 unites of electricity respectively, while u and v uses 86.2 and 110.1 unites of electricity respectively.
Suppose we can send the unites of electricity from the generation plants to the cities, and the costs are:

- from α to u is x_1 at the cost of 31.7$ per unit,
- from α to v is x_2 at the cost of 28.6$ per unit,
- from β to u is x_3 at the cost of 17.6$ per unit,
- from β to v is x_4 at the cost of 37.4$ per unit,
- from γ to u is x_5 at the cost of 22.8$ per unit,
- from γ to v is x_6 at the cost of 29.7$ per unit.

Hence, we can form our optimization problem as:

$$\begin{aligned} \min\ & 31.7x_1 + 28.6x_2 + 17.6x_3 + 37.4x_4 + 22.8x_5 + 29.7x_6, \\ \text{s.t. } & x_1 + x_2 \le 65.2, \qquad x_3 + x_4 \le 98.6, \qquad x_5 + x_6 \le 32.5, \\ & x_1 + x_3 + x_5 \ge 86.2, \qquad x_2 + x_4 + x_6 \ge 110.1, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0. \end{aligned}$$

It can be noted that all the inequalities except the last **cannot** be strict, since $65.2 + 98.6 + 32.5 = 86.2 + 110.1$. Hence, we can turn the problem into the following:

$$\begin{aligned} \min\ & 31.7x_1 + 28.6x_2 + 17.6x_3 + 37.4x_4 + 22.8x_5 + 29.7x_6, \\ \text{s.t. } & x_1 + x_2 = 65.2, \qquad x_3 + x_4 = 98.6, \qquad x_5 + x_6 = 32.5, \\ & x_1 + x_3 + x_5 = 86.2, \qquad x_2 + x_4 + x_6 = 110.1, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0. \end{aligned}$$

We equivalently write it as:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t. } & Ax = b, \\ & x \ge 0, \end{aligned}$$

where $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix}$, $b = \begin{pmatrix} 65.2 \\ 98.6 \\ \vdots \\ 110.1 \end{pmatrix}$ and $c = \begin{pmatrix} 31.7 \\ 28.6 \\ 29.7 \end{pmatrix}$. ◊

Remark II.1.5. A linear program in standard form would be portrayed as:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t. } & Ax = b, \\ & x \ge 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, and the inequality here is *coordinate*-wise. ⌟

Readers should investigate how to transfer between linear programs in standard and canonical forms just like in Remark I.1.3, we will develop a *simplex* algorithm in Section II.6.

II.2 Converting Between Linear Programing Forms

There are some very easy examples in changing between some aspects of linear programing forms.

Example II.2.1. To turn the maximum into a minimum problem, we can just negate the objective function:

- For example, $\max 6x_1 + 3x_2 - 8x_3 \cong \min -6x_1 - 3x_2 + 8x_3$.

To convert between $\geq$ and $\leq$, we have $Ax \geq b \iff -Ax \leq -b$.

- For example, $6x_1 + 3x_2 - 8x_3 \leq 7 \iff \min -6x_1 - 3x_2 + 8x_3 \geq 7$.

To convert *from* the standard form to canonical form, we have:

$$\begin{array}{ll} \min & c^\mathsf{T} x, \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{array} \iff \begin{array}{ll} \min & c^\mathsf{T} x, \\ \text{s.t.} & Ax \geq b, \\ & -Ax \geq -b, \\ & x \geq 0. \end{array} \iff \begin{array}{ll} \min & c^\mathsf{T} x, \\ \text{s.t.} & \begin{pmatrix} A \\ -A \end{pmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix}, \\ & x \geq 0. \end{array}$$

◊

The above cases are rather simple, and we will now discuss a more complicated case.

Example II.2.2. Converting Canonical Form to Standard Form.
Consider the problem in canonical form only subjected to inequalities:

$$\begin{array}{c} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \geq b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \geq b_2, \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \geq b_m, \\ x_1, x_2, \cdots, x_n \geq 0. \end{array} \iff \begin{array}{c} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_{n+1} = b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n - x_{n+2} = b_2, \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - x_{2n} = b_m, \\ x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{2n} \geq 0. \end{array}$$

Thus, by adding **slack variables** $x_{n+1}, \cdots, x_{2n}$, we are able to convert inequalities to equalities by absorbing the slack variables. ◊

If we would like to write it in the matrix form, we have:

$$\begin{array}{ll} \min & c^\mathsf{T} x, \\ \text{s.t.} & Ax \geq b, \\ & x \geq 0. \end{array} \iff \begin{array}{ll} \min & c^\mathsf{T} x, \\ \text{s.t.} & Ax - z = b, \\ & x \geq 0, z \geq 0. \end{array} \iff \begin{array}{ll} \min & \begin{pmatrix} c \\ 0 \end{pmatrix}^\mathsf{T} \begin{pmatrix} x \\ z \end{pmatrix}, \\ \text{s.t.} & \begin{pmatrix} A & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b, \\ & \begin{pmatrix} x \\ z \end{pmatrix} \geq 0. \end{array}$$

Which we effectively used matrix concatenation process.

Remark II.2.3. By convention, we consider slack variable only if we *artificially* create a new variable. ⌟

Then, we will incooperate a few other methods in getting a more standard form.

Example II.2.4. We may absorb the nonnegative constants into the matrix:

$$\begin{array}{c} Ax \geq b \\ x \geq 0 \end{array} \iff \begin{pmatrix} A \\ \mathrm{Id} \end{pmatrix} x \geq \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Similarly, for unconstrained signs, we can also make nonnegative variables:

$$\begin{array}{ll} \min & c^\intercal x, \\ \text{s.t.} & Ax \geq b. \end{array} \iff \begin{array}{ll} \min & c^\intercal (x-z), \\ \text{s.t.} & A(x-z) = b, \\ & x \geq 0, z \geq 0. \end{array}$$

◊

As canonicals of linear algebra about system of linear equations, we recall the following row operations:

(i) Swap the order of the equation.

(ii) Multiply an equation by a nonzero scalar.

(iii) Add one equation to another.

Note that these operations will not impact the solution set, which is called row-equivalent.

Example II.2.5. Consider x_1, x_2, x_3 as variables, and consider:

$$\begin{cases} 2x_1 - 2x_2 + 6x_3 = 8, \\ 3x_1 + x_2 - x_3 = 2. \end{cases} \cong \begin{cases} x_1 - x_2 + 3x_3 = 4, \\ 3x_1 + x_2 - x_3 = 2. \end{cases} \cong \begin{cases} x_1 - x_2 + 3x_3 = 4, \\ 4x_2 - 10x_3 = -10. \end{cases}$$

$$\cong \begin{cases} x_1 - x_2 + 3x_3 = 4, \\ x_2 - \frac{5}{2}x_3 = -\frac{5}{2}. \end{cases} \cong \begin{cases} x_1 + \frac{1}{2}x_3 = \frac{3}{2}, \\ x_2 - \frac{5}{2}x_3 = -\frac{5}{2}. \end{cases}$$

which is equivalently by doing the row-reduced process. ◊

Now, we recall some properties from linear algebra.

Proposition II.2.6. Suppose $E, F \in \mathbb{R}^{m\times n}$, we have $E \underset{\text{equiv.}}{\overset{\text{row}}{\sim}} F$ if and only if there exists invertible matrix $G \in \mathbb{R}^{m\times m}$ such that $GE = F$.

Sometimes, it is also useful to change order.

Example II.2.7. Consider the equivalence of the following two equations:

$$\begin{array}{ll} \min & \begin{pmatrix} 17 \\ 83 \\ -11 \end{pmatrix}^\intercal \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \text{s.t.} & \begin{pmatrix} 2 & -2 & 6 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \\ & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq 0 \end{array} \iff \begin{array}{ll} \min & \begin{pmatrix} 83 \\ -11 \\ 17 \end{pmatrix}^\intercal \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} \\ \text{s.t.} & \begin{pmatrix} -2 & 6 & 2 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \\ & \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} \geq 0 \end{array}$$

We are doing this without loss of generality for $A = \begin{pmatrix} B \mid N \end{pmatrix}$. ◊

II.3 Structure of Feasible Region

We will first review a few concepts from linear algebra.

Remark II.3.1. Inner Product.
Suppose $p, x \in \mathbb{R}^n$, the **inner product** of p and x satisfies that:

$$p^\intercal x = \|p\| \cdot \|x\| \cos\theta.$$

Note that the angle is conceptual in $\mathbb{R}^2$ and $\mathbb{R}^3$, but in fact, the angle in higher $\mathbb{R}^n$ are defined using inner products. ⌟

In particular, we can illustrate the angle as:

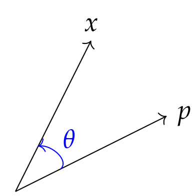


Figure II.1. Angle θ between two vectors.

There, we consider:

$$p^\intercal x \begin{cases} > 0, & \text{if } \theta \text{ is acute,} \\ = 0, & \text{if } p \text{ and } x \text{ are orthogonal,} \\ < 0, & \text{if } \theta \text{ is acute.} \end{cases}$$

Definition II.3.2. Suppose $p \in \mathbb{R}^n$ is nonzero, we defined:

$$\{x \in \mathbb{R}^n : p^\intercal x = 0\}$$

as the hyperplane through the origin with normal vector p, and:

$$\{x \in \mathbb{R}^n : p^\intercal x \geq 0\} \qquad \text{and} \qquad \{x \in \mathbb{R}^n : p^\intercal x \leq 0\}$$

are the (closed) half space with respect to p. ⌟

Remark II.3.3. With the exact same set up, we have:

- The hyperplane through the origin with normal vector p is a vector field of dimension $n - 1$.
- The two half spaces both contains the hyperplane through the origin with normal vector p, so they are both closed.

- The two half spaces are not vector spaces, but their dimension is still n.

Definition II.3.4. Suppose $p \in \mathbb{R}^n$ is nonzero and $z \in \mathbb{R}^n$, we have:

$$\{x \in \mathbb{R}^n : p^\intercal(x - z) = 0\}$$

being the hyperplane through z with normal vector p, and with half planes through z similarly defined as:

$$\{x \in \mathbb{R}^n : p^\intercal(x - z) \geq 0\} \qquad \text{and} \qquad \{x \in \mathbb{R}^n : p^\intercal(x - z) \leq 0\}. \tag{1}$$

Here, we note that $p^\intercal(x - z) = p^\intercal x - p^\intercal z$ and $p^\intercal z$ is a constant. Therefore, we can write (1) as:

$$\{x \in \mathbb{R}^n : p^\intercal x \geq \alpha\} \qquad \text{for some } \alpha \in \mathbb{R}.$$

Definition II.3.5. Polyhedron Set.
A **polyhedron** is the intersection of finitely many half spaces.

Note that the finite intersection of closed sets are still closed so polyhedron must be closed.

Example II.3.6. Suppose $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^m$, $\{x \in \mathbb{R}^n : Ax \geq b\}$ is a polyhedron, since we can consider A as row vectors:

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} x \geq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

and hence, we have x satisfying $Ax \geq b$ is equivalent to:

$$A_1^\intercal x \geq b_1, \qquad A_2^\intercal x \geq b_2, \qquad \cdots, \qquad A_m^\intercal x \geq b_m.$$

Thus, x must simultaneously be in m half spaces, *i.e.*, the intersection of the m half spaces. ♢

Definition II.3.7. Suppose $\alpha \in \mathbb{R}$, an α-level set is defined to be:

$$\{x \in \mathbb{R}^n : f(x) = \alpha\}.$$

Proposition II.3.8. If the linear program has an extreme, there must always exists an extreme appearing on a corner.

Example II.3.9. Let an optimization problem be constructed as follows:

$$\begin{aligned} \min\ & x_1 - 3x_2, \\ \text{s.t. } & x_1 - 2x_2 \geq -6, \\ & x_1 + x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can quite easily draw the feasible region in $\mathbb{R}^2$:

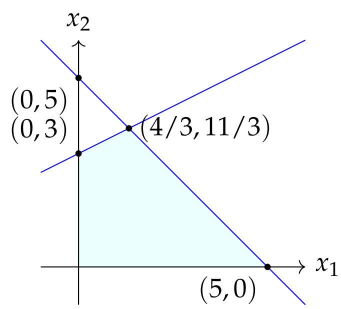


Figure II.2. Feasible region of the above problem with corners.

Here, we notice that we can format the α-level set as $x_1 - 3x_2 = \alpha$, or $x_2 = \frac{1}{3}x_1 - \frac{\alpha}{3}$.

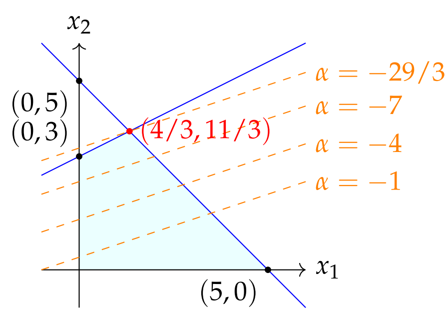


Figure II.3. Feasible region of the above problem with corners and α-level sets.

It can be noted that the minimum is reached at $x_1 = 4/3$ an $x_2 = 11/3$. ♢

II.4 Convex Sets

Definition II.4.1. Convex Combination.
Let $x, y \in \mathbb{R}^n$, a convex combination of x, y is $\lambda x + (1 - \lambda)y$ where $0 \leq \lambda \leq 1$. ⌟

One can consider the convex combination as the "connection line" from x to y, since we can write:

$$\lambda x + (1 - \lambda)y = y + \lambda(x - y),$$

so we can visualize the vector as:

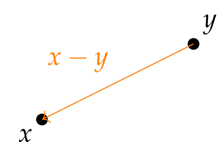


Figure II.4. Convex combination of two points in terms of a vector.

Definition II.4.2. Given k points $x^{(1)}, x^{(2)}, \cdots, x^{(k)} \in \mathbb{R}^n$, a convex combination of them is:

$$\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_k x^{(k)}, \qquad \text{where } \lambda_1, \lambda_2, \cdots, \lambda_k \geq 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1.$$

⌟

Example II.4.3. Consider $x = (1, 1)$, $y = (4, 0)$, and $z = (3, 2)$, we can form some functions such as:

$$\frac{1}{4}x + \frac{3}{4}y + 0z = (3.25, 0.25) \qquad \frac{1}{2}x + \frac{1}{4}y + \frac{1}{4}z = (2.25, 1).$$

♢

Definition II.4.4. Convex Set.
A set $S \subset \mathbb{R}^n$ is convex precisely when for all $x, y \in S$ and for all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in S$. ⌟

Proposition II.4.5. Every polyhedron $S = \{x \in \mathbb{R}^n : Ax \geq b\}$ is convex.

Proof. For any $x, y \in S$ and $\lambda \in [0, 1]$, we have $Ax \geq b$ and $Ay \geq b$, then $A\big(\lambda x + (1 - \lambda)y\big) = \lambda Ax + (1 - \lambda)Ay \geq \lambda b + (1 - \lambda)b = b$. Hence, $\lambda x + (1 - \lambda)y \in S$. □

Proposition II.4.6. $S \subset \mathbb{R}^n$ is convex if and only if every convex combination of every finite subset of S is in S.

Definition II.4.7. Basic Feasible Vector.
A vector $\hat{x} \in S$ is a **basic feasible vector** if the columns of A associated to nonzero $\hat{x}$ entry is linearly independent. ⌟

Definition II.4.8. Extreme Point.
Let $S \subset \mathbb{R}^n$ be a convex set, $x \in S$ is an **extreme point** of S precisely when $x = \lambda y + (1 - \lambda)z$ for $y, z \in S$ and $\lambda \in (0, 1)$ implies $x = y = z$. ⌟

Now, we consider the following example.

Example II.4.9. Consider the problem:

$$\begin{aligned}\min\ & 10x_1+10x_2-3x_4-x_5-3x_6,\\ \text{s.t}\ & -x_1+2x_2+3x_3+6x_4+9x_5+8x_6=26,\\ & -2x_1+3x_2+x_3+x_4+6x_5+8x_6=17,\\ & x_1+x_2-x_3+x_4+x_5+3x_6=1,\\ & x_1,x_2,x_3,x_4,x_5,x_6\geq 0.\end{aligned}$$

We temporarily suppose that $x_4=x_5=x_6=0$, then the system becomes:

$$\underbrace{\begin{pmatrix}-1 & 2 & 3\\ -2 & 3 & 1\\ 1 & 1 & -1\end{pmatrix}}_{B}\underbrace{\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}}_{x_N}=\underbrace{\begin{pmatrix}26\\ 17\\ 1\end{pmatrix}}_{b}.$$

Here, we note good luck such that B is invertible. Here, we just realize that:

$$B^{-1}=\begin{pmatrix}\frac{4}{13} & -\frac{3}{13} & \frac{7}{13}\\ \frac{1}{13} & \frac{2}{13} & \frac{5}{13}\\ \frac{5}{13} & -\frac{3}{13} & -\frac{1}{13}\end{pmatrix}$$

Thus, we have $x_B=B^{-1}b=\begin{pmatrix}2\\ 5\\ 6\end{pmatrix}$, so we temporarily have $x=(2,5,6,0,0,0)$ which, luckily, is nonnegative, so this x lies in the *basic feasible region*.

Suppose that we have chosen $x_4=\frac{1}{10}$, $x_5=\frac{1}{10}$, and $x_6=\frac{1}{5}$, so we have:

$$\begin{pmatrix}-1 & 2 & 3\\ -2 & 3 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}26\\ 17\\ 1\end{pmatrix}-\begin{pmatrix}6 & 9 & 8\\ 1 & 6 & 8\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}\frac{1}{10}\\ \frac{1}{10}\\ \frac{1}{5}\end{pmatrix}=\begin{pmatrix}\frac{229}{10}\\ \frac{147}{10}\\ \frac{1}{5}\end{pmatrix}.$$

This would end up with:

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=B^{-1}\begin{pmatrix}\frac{229}{10}\\ \frac{147}{10}\\ \frac{1}{5}\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\ \frac{41}{10}\\ \frac{27}{5}\end{pmatrix}.$$

Note that this solution nonnegative, but is not a basic feasible vector, since we have not chosen 0 basic entries.

Hence, while we need to choose certain variables as 0, we also need luck to: (i) have the matrix B being invertible (highly likely), and $x_b\geq 0$ (less likely). ♢

In general, let the problem be:

$$\begin{aligned} &\min c^\intercal x, \\ &\text{s.t. } Ax = b, \\ &x \geq 0. \end{aligned}$$

Say we have $A = \begin{pmatrix} B \mid N \end{pmatrix}$, where we suppose $B \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times(n-m)}$, so we split $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, where $x_B \in \mathbb{R}^m$ and $x_N \in \mathbb{R}^{n-m}$, so we have:

$$Bx_B + Nx_N = b \iff Bx_B = b - Nx_N \xrightarrow{\text{luck 1}} x_N = B^{-1}(b - Nx_N) \xrightarrow{\text{luck 2}} \forall x_N \geq 0 \text{ if } x_N \geq 0.$$

Example II.4.10. Continued from Example II.4.9...
We choose the temporary vector as $x_1 = x_2 = x_5 = 0$, we notice that we have:

$$\begin{pmatrix} 3 & 6 & 8 \\ 1 & 1 & 8 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 26 \\ 17 \\ 1 \end{pmatrix}.$$

This matrix B is also invertible, say $B^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{2}{13} & \frac{-8}{13} \\ \frac{11}{65} & -\frac{1}{65} & \frac{16}{65} \\ -\frac{2}{65} & \frac{9}{65} & \frac{3}{65} \end{pmatrix}$, so we have $x_B = \begin{pmatrix} 4 \\ \frac{1}{5} \\ \frac{8}{5} \end{pmatrix}$, which is also positive, and a **basic feasible vector**.
However, if we choose $x_1 = \frac{1}{5}$, $x_2 = \frac{3}{5}$, and $x_5 = 1$, we have:

$$\begin{pmatrix} 3 & 6 & 8 \\ 1 & 1 & 8 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 26 \\ 17 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 & 2 & 9 \\ -2 & 3 & 6 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} \\ \frac{3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ \frac{48}{5} \\ -\frac{4}{5} \end{pmatrix}.$$

Now, we have $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{16}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}$ so it is in the feasible region, but not basic feasible vector. ◊

Let the optimization problem be:

$$\begin{aligned} &\min c^\intercal x, \\ &\text{s.t. } Ax = b, \\ &x \geq 0. \end{aligned}$$

Now, suppose that $A \in \mathbb{R}^{m\times n}$, without loss of generality, we suppose that $\dim(\operatorname{im} A) = m$, *i.e.*, full row rank, since we can otherwise remove a reduced row of 0's.
We consider A as column vectors and we can find some inverse with respect to x_B.

Theorem II.4.11. Let $A \in \mathbb{R}^{m\times n}$ has $\dim(\operatorname{im} A) = m$, and let $b \in \mathbb{R}^m$. Say $S := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. Then $\hat{x}$ is an extreme point of S if and only if $\hat{x}$ is a basic feasible vector.

Proof. ($\Longleftarrow$:) Suppose $\hat{x}$ is a basic feasible vector. Without loss of generality, we have $A = \begin{pmatrix} B \mid N \end{pmatrix}$ since we can otherwise rearrange the vectors, such that $B \in \mathbb{R}^{m\times m}$ is the basis that is invertible, and $\hat{x} = \begin{pmatrix} \hat{x}_B \\ \hat{x}_N \end{pmatrix}$, in which $\hat{x}_B = B^{-1}b$ and $\hat{x}_N = 0$.
Suppose there exists $x', x'' \in S$ and $\lambda \in (0,1)$ such that $\hat{x} = \lambda x' + (1-\lambda)x''$. In particular:

$$0 = \hat{x}_N = \underbrace{\lambda}_{>0}\underbrace{x'_N}_{\geq 0} + \underbrace{(1-\lambda)}_{>0}\underbrace{x''_N}_{\geq 0},$$

hence we must have $x'_N = x''_N = 0$.
Hence, $Bx'_B = Bx'_B + Nx'_N = Ax' = b$, hence, we have $x'_B = B^{-1}b$. The same applies for x''_B in which $x''_B = B^{-1}b$.
Thus, $x_B = x'_B = x''_B$, which pushes $\hat{x} = x' = x''$. Hence, $\hat{x}$ is an extreme point of S.
($\Longrightarrow$:) Suppose $\hat{x} \in S$ is not a basic feasible vector, then $\{A_i : x_i > 0\}$ are linearly dependent, hence, there exists a nonzero vector $z \in \mathbb{R}^n$ such that $Az = 0$ and $z_i \neq 0$ only if $\hat{x}_i > 0$. Hence, there exists some $\delta > 0$ small enough such that:

$$x' := \hat{x} + \delta z \geq 0 \qquad \text{and} \qquad x'' := \hat{x} - \delta z \geq 0.$$

Note that x' and x'' are in S since we have:

$$Ax' = A(\hat{x} + \delta z) = A\hat{x} + \delta Az = A\hat{x} = b \qquad \text{and} \qquad x' \geq 0,$$

while:

$$Ax'' = A(\hat{x} - \delta z) = A\hat{x} - \delta Az = A\hat{x} = b \qquad \text{and} \qquad x'' \geq 0.$$

Hence, by construction, we have:

$$\frac{1}{2}x' + \frac{1}{2}x'' = \hat{x},$$

hence $\hat{x}$ is not extreme point of S. □

II.5 Objective Function, Revisit

Consider the linear programing problem:

$$\begin{aligned} \min\ & c^\mathsf{T} x, \\ \text{s.t. } & Ax = b, \\ & x \geq 0. \end{aligned}$$

Without loss of generality, we suppose $A \in \mathbb{R}^{m\times n}$ as the matrix of full row rank. Let's say $A = \begin{pmatrix} B & N \end{pmatrix}$ and $B \in \mathbb{R}^{m\times m}$ is invertible (which is the basis for a basic feasible vector). We consider $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, in which we consider $Bx_B + Nx_N = Ax = b$. Hence, if we take $\hat{x_n} = 0$, then we have $\hat{x}_B = B^{-1}b$ (which we hope $\hat{x}_B \geq 0$) , so we have the basic feasible vector as $\hat{x} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$.

Now, if we choose any $x_N \geq 0$, for $x_B = B^{-1}(b - Nx_N)$. If $x \geq 0$, then $\begin{pmatrix} x_B \\ x_N \end{pmatrix}$ is feasible.

Now, we bring in $c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$ with the object function defined as:

$$\begin{aligned} c^\intercal x &= \begin{pmatrix} c_B \\ c_N \end{pmatrix}^\intercal \begin{pmatrix} x_B \\ x_N \end{pmatrix} = c_B^\intercal x_B + c_N^\intercal x_N \\ &= c_B^\intercal B^{-1}(b - Nx_N) + c_N^\intercal x_N = \underbrace{c_B^\intercal B^{-1} b}_{\substack{\text{objective function value} \\ \text{at basic feasible vector}}} + \underbrace{(c_N^\intercal - c_B^\intercal B^{-1} N)}_{\substack{r_N^\intercal \\ \text{reduced cost}}} x_N, \end{aligned}$$

which is called the reduced cost.

Example II.5.1. Now, recall one of the old example:

$$\begin{aligned} \min \; & \begin{pmatrix} 10 & 10 & 0 & -3 & -1 & -3 \end{pmatrix} x, \\ \text{s.t.} \; & \begin{pmatrix} -1 & 2 & 3 & 6 & 9 & 8 \\ -2 & 3 & 1 & 1 & 6 & 3 \\ 1 & 1 & -1 & 1 & 1 & 8 \end{pmatrix} x = \begin{pmatrix} 26 \\ 17 \\ 1 \end{pmatrix}, \\ & x \geq 0. \end{aligned} \tag{2}$$

Hence, we have:

$$B = \begin{pmatrix} -1 & 2 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

and so we have $B^{-1}b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$ with the basic feasible vector as $\begin{pmatrix} 2 \\ 5 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, and we now have:

$$\begin{aligned} r_N^\intercal = c_N^\intercal - c_B^\intercal B^{-1} N &= \begin{pmatrix} -3 & -1 & -3 \end{pmatrix} - \begin{pmatrix} 10 & 10 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{13} & -\frac{3}{13} & \frac{7}{13} \\ \frac{1}{13} & \frac{2}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{3}{13} & -\frac{1}{13} \end{pmatrix} \begin{pmatrix} 6 & 9 & 8 \\ 1 & 6 & 3 \\ 1 & 1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} -33 & -31 & -43 \end{pmatrix}. \end{aligned}$$

So we have the objective function with reduced cost as:

$$\text{reduced cost objective function} = 70 - 33x_4 - 31x_5 - 43x_6. \tag{3}$$

In fact, since all coefficients are negative, we do get a global maximum though. However, we want a minimum for this problem.

Note that we can move these x_4, x_5, x_6 above a little bit so we can have smaller objective values, however, we need to consider the constraint that $x_B \geq 0$, since when x_N is arbitrarily big, $B^{-1}(b - Nx_N)$ can be negative.

Now, consider picking another basic entries 3, 4, and 6, we then have:

$$B = \begin{pmatrix} 3 & 6 & 8 \\ 1 & 1 & 3 \\ -1 & 1 & 8 \end{pmatrix},$$

and so we have $B^{-1}b = \begin{pmatrix} 4 \\ \frac{1}{5} \\ \frac{8}{5} \end{pmatrix}$ with the basic feasible vector as $\begin{pmatrix} 0 \\ 0 \\ 4 \\ \frac{1}{5} \\ 0 \\ \frac{8}{5} \end{pmatrix}$, and we now have:

$$r_N^\intercal = c_N^\intercal - c_B^\intercal B^{-1}N = \begin{pmatrix} 10 & 10 & -1 \end{pmatrix} - \begin{pmatrix} 0 & -3 & -3 \end{pmatrix} \begin{pmatrix} \frac{1}{13} & \frac{2}{13} & -\frac{8}{13} \\ \frac{11}{65} & -\frac{17}{65} & \frac{16}{65} \\ -\frac{2}{65} & \frac{9}{65} & \frac{3}{65} \end{pmatrix} \begin{pmatrix} -1 & 2 & 9 \\ -2 & 3 & 6 \\ 1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{56}{5} & \frac{53}{5} & \frac{7}{5} \end{pmatrix}.$$

So we have the objective function value:

$$\text{ofv} = -\frac{27}{5} + \frac{56}{5}x_1 + \frac{53}{5}x_2 - \frac{7}{5}x_5. \tag{4}$$

Note that since $r_N^\intercal \geq 0$, this implies that taking $x_N = 0$ is the best.

We can check that these ofv are corresponding to each other. We plug in $x_1 = 2$, $x_2 = 5$, and $x_5 = 0$ to (4):

$$-\frac{27}{5} + \frac{56}{5} \cdot 2 + \frac{53}{5} \cdot 5 + \frac{7}{5} \cdot 0 = 70,$$

where as if we plug in $x_4 = \frac{1}{5}$, $x_5 = 0$, and $x_6 = \frac{8}{5}$ to (3):

$$70 - 33 \cdot \frac{1}{5} - 31 \cdot 0 - 43 \cdot \frac{8}{5} = -\frac{27}{5}.$$

Here, diligent readers should notice that the $r_N^\intercal$ tells us how deviations of nonbasic entries will react to small changes. ♢

Remark II.5.2. Recall in Theorem II.4.11, the extreme points must be basic feasible vector and we would be seeking for a picking of vectors such that $r_N^\intercal$ has all its entries positive or negative. ⌟

II.6 Simplex Algorithm

Again, we will recall the setup in (2), so we can form a **simplex pre-tableau**, where we can rewrite the problem as:

1	-10	-10	0	3	1	3	0
0	-1	2	3	6	9	8	26
0	-2	3	1	1	6	8	17
0	1	1	-1	1	1	3	1

Table II.5. Simplex pre-tableau for (2).

Remark II.6.1. The first column can be considered as a new variable z, so the first row defines that:

$$z - 10x_1 - 10x_2 + 0x_3 + 3x_4 + 1x_5 + 3x_6 = 0,$$

so that we have z assigned to be the objective function (cost). ⌟

Then, we are about to create a new tableau correspondingly with row reduction:

- First, with all the rows reduced with the first nonheader row:

1	0	-30	-30	-57	-89	-77	-260
0	1	-2	-3	-6	-9	-8	-26
0	0	-1	-5	-11	-12	-8	-35
0	0	3	2	7	10	11	27

Table II.6. Simplex pre-tableau manipulation for (2): step 1.

- Then, we reduce with the second nonheader row:

1	0	0	120	273	271	163	790
0	1	-2	-3	-6	-9	-8	-26
0	0	1	5	11	12	8	35
0	0	0	-13	-26	-26	-13	-78

Table II.7. Simplex pre-tableau manipulation for (2): step 2.

- Then, we manipulate the last row:

1	0	0	0	33	31	43	70
0	1	-2	-3	-6	-9	-8	-26
0	0	1	5	11	12	8	35
0	0	0	1	2	2	1	6

Table II.8. Simplex pre-tableau manipulation for (2): step 3.

- Eventually, we clean up the first three columns:

1	0	0	0	33	31	43	70
0	1	0	0	2	1	1	2
0	0	1	0	1	2	3	5
0	0	0	1	2	2	1	6

Table II.9. Simplex pre-tableau manipulation for (2): step 4.

Now, we have obtained that Table II.9 is the basic feasible tableau, which we can rewrite the first row as:

$$z = 70 - 33x_4 - 33x_5 - 43x_6,$$

which corresponds to (3).

Remark II.6.2. We can notice that the first three columns of the header row are zeros, and thee first three columns of the nonheader row forms the identity matrix, while the right hand side is nonnegative. ⌟

Also, note that the last column is not guaranteed to be nonnegative. However, we have nothing to deal with it up to right now. (Will appear later about *duality* at Section III.3.)

Given our setup that:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t. } & Ax = b, \\ & x \geq 0, \end{aligned}$$

we have:

$$\begin{array}{c|c|c} 1 & -c^\intercal & 0 \\ \hline 0 & A & b \end{array} \leftrightarrows \begin{array}{c|cc|c} 1 & -c_B^\intercal & -c_N^\intercal & 0 \\ \hline 0 & B & N & b \end{array} \xleftarrow[\substack{B \text{ is invertible} \\ B^{-1}b \geq 0}]{} \begin{array}{c|cc|c} 1 & 0^\intercal & -c_N^\intercal + c_B^\intercal B^{-1}N & c_B^\intercal B^{-1}b \\ \hline 0 & \mathrm{Id} & B^{-1}N & B^{-1}b \end{array}$$

Table II.10. Basic level understanding of simplex pretableau.

Here, in the last step, we have:

$$z = c_B^\intercal B^{-1} v + r_N^\intercal x_N.$$

Remark II.6.3. For the following steps, we would use the *greedy approach* to pivot. ⌟

Hence, we try with pivoting 43 first, *i.e.*, manipulating x_1, we retrieve the following relationship from Table II.9:

$$\begin{aligned} x_1 &= 2 - 1x_6 && \text{when } x_6 > 2,\ x_1 \text{ will go negative.} \\ x_2 &= 5 - 3x_6 && \text{when } x_6 > \tfrac{5}{3},\ x_2 \text{ will go negative.} \\ x_3 &= 6 - 1x_6 && \text{when } x_6 > 6,\ x_3 \text{ will go negative.} \end{aligned}$$

Thus, x_6 can at most go to $\frac{5}{3}$. (Note if any of the coefficients is negative, in fact, we can just ignore this algorithm.)

Effectively, we would have $x_2 = 0$, and x_6 becomes a part of the basis.

Thereby, we now manipulate on column 6 row 2:

1	0	$-\frac{43}{3}$	0	$\frac{56}{3}$	$\frac{7}{3}$	0	$-\frac{5}{3}$
0	1	$-\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
0	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{5}{3}$
0	0	$-\frac{1}{3}$	1	$\frac{5}{3}$	$\frac{4}{3}$	0	$\frac{13}{3}$

Table II.11. Simplex tableau manipulation for (2) with x_6 becoming basic vector.

Here, Table II.11 is still simplex tableau, where we have obtained the basic feasible vector:

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{13}{3} & 0 & 0 & \frac{5}{3} \end{pmatrix}^\intercal .$$

Here, we notice the largest cost as $\frac{56}{3}$ and we will compute the relationship in Table II.11:

$$\begin{aligned} x_1 &= \tfrac{1}{3} - \tfrac{5}{3}x_4 && \text{when } x_4 > \tfrac{1}{5}, x_1 \text{ will go negative.} \\ x_3 &= \tfrac{5}{3} - \tfrac{1}{3}x_4 && \text{when } x_4 > 5, x_3 \text{ will go negative.} \\ x_6 &= \tfrac{13}{3} - \tfrac{5}{3}x_4 && \text{when } x_4 > \tfrac{13}{5}, x_5 \text{ will go negative.} \end{aligned}$$

Therefore, we will be manipulating on x_1, so the table becomes:

1	$-\frac{56}{5}$	$-\frac{53}{5}$	0	0	$-\frac{7}{5}$	0	$-\frac{27}{5}$
0	$\frac{3}{5}$	$-\frac{1}{5}$	0	1	$\frac{1}{5}$	0	$\frac{1}{3}$
0	$-\frac{1}{5}$	$\frac{2}{5}$	0	0	$\frac{3}{5}$	1	$\frac{5}{3}$
0	-1	0	1	0	1	0	$\frac{13}{3}$

Table II.12. Simplex tableau manipulation for (2) with x_4 becoming basic vector.

It is easy to notice that Table II.12 is a simplex tableau, and the basic feasible vector:

$$\begin{pmatrix} 0 & 0 & 4 & \frac{1}{5} & 0 & \frac{8}{3} \end{pmatrix}^\intercal .$$

More importantly, we have noticed that all entries in the middle of the header row is negative, which implies that we have reached the **optimal solution**.

Definition II.6.4. Degeneracy and Stalling.
Degeneracy occurs when the basic entires of the basic feasible vector (right hand side column) has a value 0. Stalling occurs when the objective function does not change after a simplex iteration. ⌟

Likewise, if we have a zero on the last column, then if the candidate has positive entry when replacing that is "stall," (if it is negative or zero, it is fine).

Note that if there is not a positive entry, degeneracy does not cause stall. However, there is a worse case.

Remark II.6.5. Absolute Degeneracy.
Through the simplex algorithm, given a column j (positive cost), we have been selecting pivoting row by:

$$z \in \underset{k, \widehat{a_{k,j}} > 0}{\arg\min} \frac{\hat{b}_k}{\widehat{a_{k,j}}}.$$

Note that there could be no k such that $\widehat{a_{k,j}}$ is positive, then this leads to **absolute degeneracy** when the function is unbounded from below. ⌟

One good property of our simplex algorithm pivoting is that it guarantees to maintain the basic feasible vector to be feasible when it used to be feasible. Now, we need to investigate how to start up a simplex pretableau with the basic matrix.

Example II.6.6. Consider the linear programming, let:

$$\begin{aligned} &\min 3x_1 + 4x_2 - x_3, \\ &\text{s.t. } 6x_1 + 9x_2 + x_3 \leq 8, \\ &\quad -x_1 + 6x_2 - 4x_3 \leq 1, \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

We would first need to put in *slack* variables to make it into standard form:

$$\begin{aligned} &\min 3x_1 + 4x_2 - x_3, \\ &\text{s.t. } 6x_1 + 9x_2 + x_3 + x_4 = 8, \\ &\quad -x_1 + 6x_2 - 4x_3 + x_5 = 1, \\ &\quad x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Now, we set the slack variables to 0 so we have:

1	−3	−4	1	0	0	0
0	6	9	1	1	0	8
0	−1	6	−4	0	1	1

Table II.13. Simplex tableau with slack variables being 0.

Directly, we are equipped with the first basic feasible vector $(0, 0, 0, 8, 1)$ and we can start pivoting with the greedy algorithm from there. ◊

Without loss of generality, we can assume the rightmost column to be positive, otherwise, we can multiple -1 on the whole row. However, the simplex tableau might not be started up nicely, then we need to somehow find the first basic feasible vector, using the method known as the **Two Phase Method**.

Example II.6.7. Now, we consider solving a problem via two phase method.

- **Phase 1**:

$$\begin{aligned} &\min 4x_1 + x_2 + x_3, \\ &\text{s.t. } 2x_1 + x_2 + 2x_3 = 4, \\ &\quad 3x_1 + 3x_2 + x_3 = 3, \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned} \qquad \begin{aligned} &\min x_4 + x_5, \\ &\text{s.t. } 2x_1 + x_2 + 2x_3 + x_4 = 4, \\ &\quad 3x_1 + 3x_2 + x_3 + x_5 = 3, \\ &\quad x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

  Then, we have the table:

1	0	0	0	-1	-1	0
0	2	1	2	1	0	4
0	3	3	1	0	1	3

Table II.14. Simplex tableau with new artificial variable.

Then, we would reduce the last two columns of artificial variables:

1	5	4	3	0	0	7
0	2	1	2	1	0	4
0	3	3	1	0	1	3

Table II.15. Simplex tableau after clearing out the costs for artificial variable.

Remark II.6.8. If we already have a slack variable serving as a canonical basis, we do not need to add in that artificial variable for that basis. ⌟

Afterwards, we now would pivot on the first row and third row to get that:

1	0	0	0	-1	-1	0
0	0	$-\frac{3}{4}$	1	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$
0	1	$\frac{5}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

Table II.16. Simplex tableau with initial entry.

Here, the artificial variables leave the basis. Therefore, we have the current basic feasible vector as $\left(\frac{1}{2}, 0, \frac{3}{2}, 0, 0\right)$.

Remark II.6.9. If the artificial variables do not leave the basis, or if the objective function value is positive in the end, this means that problem is not feasible. ⌟

Hence, we can assume they leave and can proceed forward when there is a feasible solution.

- **Phase 2**: First, we consider plugging in the original problem, namely:

1	-4	-1	-1	0
0	0	$-\frac{3}{4}$	1	$\frac{3}{2}$
0	1	$\frac{5}{4}$	0	$\frac{1}{2}$

Table II.17. Simplex tableau before pivoting.

Again, we clear out the cost function above, and we then pivot with the second row, from:

1	0	$\frac{13}{4}$	0	$\frac{7}{2}$
0	0	$-\frac{3}{4}$	1	$\frac{3}{2}$
0	1	$\frac{5}{4}$	0	$\frac{1}{2}$

Table II.18. Simplex tableau before pivoting.

Therefore, we have the simplex tableau:

1	$-\frac{13}{5}$	0	0	$\frac{11}{5}$
0	$\frac{3}{4}$	0	1	$\frac{9}{5}$
0	$\frac{4}{5}$	1	0	$\frac{2}{5}$

Table II.19. Simplex tableau after pivoting.

Therefore, the basic feasible vector is now $\left(0, \frac{2}{5}, \frac{9}{5}\right)$. ♢

In general, for the two-phase method, for phase 1, we would consider the problem that:

$$\min \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} x \\ y \end{pmatrix},$$

such that we make up the constraints that:

$$\begin{pmatrix} A & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = b$$

for $b \geq 0$ and $\begin{pmatrix} x \\ y \end{pmatrix} \geq 0$.

Then, for phase 2, we may consider the original columns of matrix and the new vector $\hat{b}$ so we can proceed by pivoting for the solution.

Remark II.6.10. The (first phase of the) two phase method also serves as an evidence that if there exists a feasible solution, then there exists a basic feasible vector. This is because that if there exists a feasible solution, then there exists a way to get rid of the artificial variables through minimizing them to 0, and hence the two phase method provides us with a basic feasible vector. ⌟

Then, we will be introducing another method, called the **Big-M Method**.

Example II.6.11. Here, we will give the artificial variables with a large cost, we turn the problem as follows:

$$\begin{aligned} \min\ & 4x_1 + x_2 + x_3, \\ \text{s.t. } & 2x_1 + x_2 + 2x_3 = 4, \\ & 3x_1 + 3x_2 + x_3 = 3, \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \qquad \begin{aligned} \min\ & 4x_1 + x_2 + x_3 + 100x_4 + 100x_5, \\ \text{s.t. } & 2x_1 + x_2 + 2x_3 + x_4 = 4, \\ & 3x_1 + 3x_2 + x_3 + x_5 = 3, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Given that 100 is a large number (well we can technically make it larger, or consider it purely as a large constant M), we can solve the problem as it was since it would try to prevent x_4 and x_5 to be positive given its large cost.

Therefore, the pre-tableu is:

1	−4	−1	−1	−100	−100	0
0	2	1	2	1	0	4
0	3	3	1	0	1	3

Table II.20. Simplex tableau with first entry.

Then, we will reduce the last two columns to cost zero, so we have:

1	496	399	299	0	0	0
0	2	1	2	1	0	4
0	3	3	1	0	1	3

Table II.21. Simplex tableau with first entry.

Then, we will pivot on these columns. Note that once the artificial variable leaves the column, we will just ignore it forever. ♢

II.7 Case Study: Sukudo Game

Recall the Sukudo game, we endow the macro/micro row/column with variables i, j, k, ℓ, m and indicator variables $x_{i,j,k,\ell} \in \{0, 1\}$ indicating if the i-th macro row, j-th macro column, k-th micro row, and ℓ-th micro column has value m if the variable is 1.

Definition II.7.1. We can parametrize the Sukudo game into several variables, portrayed as the following table:

$x_{1,1,1,1,*}$	$x_{1,1,1,2,*}$	$x_{1,1,1,3,*}$	$x_{1,2,1,1,*}$	$x_{1,2,1,2,*}$	$x_{1,2,1,3,*}$	$x_{1,3,1,1,*}$	$x_{1,3,1,2,*}$	$x_{1,3,1,3,*}$
$x_{1,1,2,1,*}$	$x_{1,1,2,2,*}$	$x_{1,1,2,3,*}$	$x_{1,2,2,1,*}$	$x_{1,2,2,2,*}$	$x_{1,2,2,3,*}$	$x_{1,3,2,1,*}$	$x_{1,3,2,2,*}$	$x_{1,3,2,3,*}$
$x_{1,1,3,1,*}$	$x_{1,1,3,2,*}$	$x_{1,1,3,3,*}$	$x_{1,2,3,1,*}$	$x_{1,2,3,2,*}$	$x_{1,2,3,3,*}$	$x_{1,3,3,1,*}$	$x_{1,3,3,2,*}$	$x_{1,3,3,3,*}$
$x_{2,1,1,1,*}$	$x_{2,1,1,2,*}$	$x_{2,1,1,3,*}$	$x_{2,2,1,1,*}$	$x_{2,2,1,2,*}$	$x_{2,2,1,3,*}$	$x_{2,3,1,1,*}$	$x_{2,3,1,2,*}$	$x_{2,3,1,3,*}$
$x_{2,1,2,1,*}$	$x_{2,1,2,2,*}$	$x_{2,1,2,3,*}$	$x_{2,2,2,1,*}$	$x_{2,2,2,2,*}$	$x_{2,2,2,3,*}$	$x_{2,3,2,1,*}$	$x_{2,3,2,2,*}$	$x_{2,3,2,3,*}$
$x_{2,1,3,1,*}$	$x_{2,1,3,2,*}$	$x_{2,1,3,3,*}$	$x_{2,2,3,1,*}$	$x_{2,2,3,2,*}$	$x_{2,2,3,3,*}$	$x_{2,3,3,1,*}$	$x_{2,3,3,2,*}$	$x_{2,3,3,3,*}$
$x_{3,1,1,1,*}$	$x_{3,1,1,2,*}$	$x_{3,1,1,3,*}$	$x_{3,2,1,1,*}$	$x_{3,2,1,2,*}$	$x_{3,2,1,3,*}$	$x_{3,3,1,1,*}$	$x_{3,3,1,2,*}$	$x_{3,3,1,3,*}$
$x_{3,1,2,1,*}$	$x_{3,1,2,2,*}$	$x_{3,1,2,3,*}$	$x_{3,2,2,1,*}$	$x_{3,2,2,2,*}$	$x_{3,2,2,3,*}$	$x_{3,3,2,1,*}$	$x_{3,3,2,2,*}$	$x_{3,3,2,3,*}$
$x_{3,1,3,1,*}$	$x_{3,1,3,2,*}$	$x_{3,1,3,3,*}$	$x_{3,2,3,1,*}$	$x_{3,2,3,2,*}$	$x_{3,2,3,3,*}$	$x_{3,3,3,1,*}$	$x_{3,3,3,2,*}$	$x_{3,3,3,3,*}$

⌟

Our goal is to describe the Sukudo table with the constraints on the variables:

- We want exactly one number at each entry, for all $i = 1, 2, 3$, $j = 1, 2, 3$, $k = 1, 2, 3$, and $\ell = 1, 2, 3$:

$$\sum_{m=1}^{9} x_{i,j,k,\ell,m} = 1.$$

- We want to have the exactly one entry at the macro block, for all $i = 1, 2, 3$, $j = 1, 2, 3$, $m = 1, \cdots, 9$:

$$\sum_{k=1}^{3} \sum_{\ell=1}^{3} x_{i,j,k,\ell} = 1.$$

- We want to have the exactly one entry at the row, for all $i = 1, 2, 3$, $k = 1, 2, 3$, $m = 1, \cdots, 9$:

$$\sum_{j=1}^{3} \sum_{\ell=1}^{3} x_{i,j,k,\ell} = 1.$$

- We want to have the exactly one entry at the column, for all $j = 1, 2, 3$, $\ell = 1, 2, 3$, $m = 1, \cdots, 9$:

$$\sum_{i=1}^{3} \sum_{k=1}^{3} x_{i,j,k,\ell} = 1.$$

Moreover, we can also propose variants with the Sukudo table:

(i) Instead of having $3 \times 3 \times 3 \times 3$ table, we can instead have a different base for each variable.

(ii) In the constraints, we restricted the for all with (i, j), (i, k), and (j, ℓ) with the sums as (k, ℓ), (j, ℓ), and (i, k). Since we are picking two as for all and the rest as sum, we would have $\binom{4}{2} = 6$ entries. The rest three constraints would be restricting the for all with (i, ℓ), (j, k), and (k, ℓ) with the sums as (j, k), (i, ℓ), and (i, j).

Here, we can formalize this into a problem. We would want to develop the optimization problem:

$$\begin{aligned} &\min c^\intercal x, \\ &\text{s.t. } Ax = b, \qquad A \in \mathbb{R}^{324, \times 729} (= 0), b \in \mathbb{R}^{324}, \\ &\quad x \in \{0, 1\}^{729}. \end{aligned}$$

For simplicity, we propose a **dictionary order**, to reenumerate the variables for x, as:

$$x = \begin{pmatrix} x_{1,1,1,1,1} & x_{1,1,1,1,2} & \cdots & x_{3,3,3,3,9} \end{pmatrix}^\intercal.$$

Definition II.7.2. Reenumeration function.
We can do a simple enumeration of this finite sequence by:

$$\iota : (i, j, k, \ell, m) \mapsto 243(i-1) + 81(j-1) + 27(k-1) + 9(\ell - 1) + m.$$

┘

Just to note, this enumeration function is one-to-one, and we can also return with an inverse function ι^{-1}.

Remark II.7.3. This is trivially true as one can consider this as a trinary number in this situation. ┘

Then, we want to fill in the numbers in the array, so the algorithm can be working out as follows:

Algorithm II.7.4. Algorithm for Constructing the Optimization Problem.

Let rowIndex $\leftarrow 0$.
for all $i \leftarrow 1 : 3$ **do**

```
        for all k ← 1 : 3 do
            for all m ← 1 : 9 do
                rowIndex ← rowIndex + 1
                for all j ← 1 : 3 do
                    for all ℓ ← 1 : 3 do
                        if x_{i,j,k,ℓ,m} = 1 then
                            A(rowIndex, ι(i, j, k, ℓ, m)) = 1.
                        end if
                    end for
                end for
            end for
        end for
    end for
```

III Duality

III.1 Dual Program

Definition III.1.1. Dual Program.
Consider $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, we can have a linear program (in canonical form, or *symmetric form*) for decision variable $x \in \mathbb{R}^n$ that:

$$\begin{aligned} \min\ & c^\top x, \\ \text{s.t. } & Ax \geq b, \\ & x \geq 0. \end{aligned} \tag{LP}$$

The **dual program** is the program for decision variable $y \in \mathbb{R}^m$ that:

$$\begin{aligned} \max\ & b^\top y, \\ \text{s.t. } & A^\top y \leq c, \\ & y \geq 0. \end{aligned} \tag{DP}$$

┘

Example III.1.2. Given a linear program that:

$$\begin{aligned} \min\ & x_1 + 2x_2, \\ \text{s.t. } & 3x_1 + 4x_2 \geq 9, \\ & 5x_1 + 6x_2 \geq 10, \\ & 7x_1 + 8x_2 \geq 11, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{LP}$$

We can form its dual program as:

$$\begin{aligned} \max\ & 9y_1 + 10y_2 + 11y_3, \\ \text{s.t. } & 3y_1 + 5y_2 + 7y_3 \leq 9, \\ & 4y_1 + 6y_2 + 8y_3 \leq 2, \\ & y_1, y_2, y_3 \geq 0. \end{aligned} \tag{DP}$$

◊

In particular, we can think of the two programs as follows:

$$\begin{matrix} (c^\top) & & & (b^\top) & \\ (A) & (b) & \leftrightharpoons & (A^\top) & (c) \\ (x^\top) & & & (y^\top) & \end{matrix}$$

Remark III.1.3. The primal and dual problem relatively feed up with each other, and it can be thought of as the lever game. ┘

Definition III.1.4. Dual Program in Standard Form.
Given a linear program (in standard form):

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t. }\ & Ax = b, \\ & x \geq 0, \end{aligned} \tag{LP}$$

where we have $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$. We define the **dual program** is the program:

$$\begin{aligned} \max\ & b^\intercal y, \\ \text{s.t. }\ & A^\intercal y \leq c, \end{aligned} \tag{DP}$$

where $y \in \mathbb{R}^m$. ⌟

Note that the dual program of the standard form does not look the same as the canonical form, but we can show that the two definitions are equivalent.

Proposition III.1.5. The dual program definitions in standard and canonical form are the same.

Proof. Recall that for the linear programming in canonical form, we have:

$$\begin{array}{ccc} \begin{array}{ll} \min & c^\intercal x, \\ \text{s.t.} & Ax \geq b, \\ & x \geq 0. \end{array} & \overset{\sim}{\longrightarrow} & \begin{array}{ll} \min & \begin{pmatrix} c \\ 0 \end{pmatrix}^\intercal \begin{pmatrix} x \\ z \end{pmatrix}, \\ \text{s.t.} & \begin{pmatrix} A\,\mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b, \\ & \begin{pmatrix} x \\ z \end{pmatrix} \geq 0. \end{array} \\ \downarrow & & \downarrow \\ \begin{array}{ll} \max & b^\intercal y, \\ \text{s.t.} & A^\intercal y \leq c, \\ & y \geq 0. \end{array} & \overset{\sim}{\longrightarrow} & \begin{array}{ll} \max & b^\intercal y, \\ \text{s.t.} & \begin{pmatrix} A^\intercal \\ -\mathrm{Id} \end{pmatrix} \leq \begin{pmatrix} x \\ 0 \end{pmatrix}. \end{array} \end{array}$$

To go from in standard form, we have:

$$\begin{array}{ccc} \begin{array}{ll} \min & c^\intercal x, \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{array} & \overset{\sim}{\longrightarrow} & \begin{array}{ll} \min & c^\intercal x, \\ \text{s.t.} & \begin{pmatrix} A \\ -A \end{pmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix}, \\ & x \geq 0. \end{array} \\ \downarrow & & \downarrow \\ \begin{array}{ll} \max & \begin{pmatrix} b^\intercal \\ -b \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \\ \text{s.t.} & \begin{pmatrix} A^\intercal & -A^\intercal \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} \leq c, \\ & \begin{pmatrix} \mu \\ \nu \end{pmatrix} \geq 0. \end{array} & \overset{\sim}{\longrightarrow} & \begin{array}{ll} \max & b^\intercal(\mu - \nu), \\ \text{s.t.} & A^\intercal(\mu - \nu) \leq c, \\ & \mu, \nu \geq 0. \end{array} \end{array}$$

□

Proposition III.1.6. The dual of the dual is the original program.

Proof. We consider, without loss of generality a canonical form, as:

$$\begin{array}{lll} \min & c^\intercal x, \\ \text{s.t.} & Ax \geq b, \\ & x \geq 0. \end{array} \xrightarrow{\text{dual}} \begin{array}{lll} \max & b^\intercal y, \\ \text{s.t.} & A^\intercal y \leq c, \\ & y \geq 0. \end{array} \overset{\sim}{\longrightarrow} \begin{array}{lll} \min & -b^\intercal y, \\ \text{s.t.} & -A^\intercal y \geq -c, \\ & y \geq 0. \end{array} \xrightarrow{\text{dual}} \begin{array}{lll} \max & -c^\intercal z, \\ \text{s.t.} & -(A^\intercal)^\intercal z \leq -b, \\ & z \geq 0. \end{array}$$

(the first and last programs are connected by $\sim$) □

Theorem III.1.7. Weak Duality.
Given LP and DP, with x feasible in LP and y feasible in DP, then:

$$\text{ofv}_{\text{DP}}(y) \leq \text{ofv}_{\text{LP}}(x), \tag{5}$$

thus $\text{oofv}_{\text{DP}} \leq \text{oofv}_{\text{LP}}$.

Proof. (For standard form:) x being feasible in LP means that $Ax = b$, $x \geq 0$. y feasible in DP means that $A^\intercal y \leq c$. Hence, we have:

$$b^\intercal y = (Ax)^\intercal y = x^\intercal A^\intercal y \leq x^\intercal c = c^\intercal x.$$

(For canonical form:) x being feasible in LP means that $Ax \geq b$, $x \geq 0$. y feasible in DP means that $A^\intercal y \leq c$. Hence, we have:

$$b^\intercal y \leq (Ax)^\intercal y = x^\intercal A^\intercal y \leq x^\intercal c = c^\intercal x.$$

□

Corollary III.1.8. Supervisor Principle.
Given LP and DP, with x feasible in LP and y feasible in DP, such that the equality holds in (5), then x is optimal in LP and y is optimal in DP.

Note that if the objective function in LP is unbounded below, then DP is infeasible. If the objective function in DP is unbounded above, then the primal LP problem is infeasible.

Remark III.1.9. The Supervisor Principle (Corollary III.1.8) serves as a quick method of verifying optimality. ⌟

Theorem III.1.10. Strong Duality.
If LP is feasible (*i.e.*, there exists a feasible vector) and DP is feasible (ibid.), then there exists $\hat{x}$ that is feasible in LP and $\hat{y}$ is feasible in DP such that $\text{ofv}_{\text{DP}}(\hat{y}) = \text{ofv}_{\text{LP}}(\hat{x})$, hence $\hat{x}$ and $\hat{y}$, respectively, are optimal in LP and DP.

Proof. Let the LP be defined as follows:

$$\begin{aligned} &\min c^\intercal x, \\ &\text{s.t. } Ax = b, \qquad x \geq 0. \end{aligned}$$

Without loss of generality, we assume that $A \in \mathbb{R}^{m\times n}$ is full rank. We run the simplex algorithm until it terminates with optimal *basic feasible vector*. Since DP is feasible, thus the objective function LP is bounded below. Say the optimal basis is B, without loss of generality with $A = \begin{pmatrix} B & N \end{pmatrix}$, and hence we have:

$$\hat{x} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}, \qquad \text{oofv}_{\text{LP}} = c_B^\intercal B^{-1} b, \qquad \text{and} \qquad r_N^\intercal \equiv c_N^\intercal - c_B^\intercal B^{-1} N \geq 0$$

which is the terminating condition. For the DP, we have:

$$\begin{aligned} &\max y^\intercal b, \\ &\text{s.t. } y^\intercal A \leq c^\intercal. \end{aligned}$$

Therefore, we consider $\hat{y}^\intercal := c_B^\intercal B^{-1}$, and thus we have $\hat{y}$ feasible in DP since:

$$\hat{y}^\intercal A = c_B^\intercal B^{-1} \begin{pmatrix} B & N \end{pmatrix} = \begin{pmatrix} c_B^\intercal & c_B^\intercal B^{-1} N \end{pmatrix} \leq \begin{pmatrix} c_B^\intercal & c_N^\intercal \end{pmatrix}.$$

Therefore, we have that:

$$\text{ofv}_{\text{DP}}(\hat{y}) = \hat{y}^\intercal b = c_B^\intercal B^{-1} b = c^\intercal \hat{x} = \text{ofv}_{\text{LP}}(\hat{x}).$$

Moreover, by the supervision principle, $\hat{x}$ and $\hat{y}$ are optimal in LP and DP respectively. □

Example III.1.11. Consider the following linear program:

$$\begin{aligned} &\min 4x_1 + x_2 + x_3, \\ &\text{s.t. } 2x_1 + x_2 + 2x_3 = 4, \\ &\quad 3x_1 + 3x_2 + x_3 = 3, \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

From the previous computation, we had the optimal *basic feasible vector* as $(0, \frac{2}{5}, \frac{9}{5})$ whose optimal objective function value is $\frac{11}{5}$.
Then, we consider the dual program as:

$$\begin{aligned} &\max 4y_1 + 3y_2, \\ &\text{s.t. } 2y_1 + 3y_2 \leq 4, \\ &\quad y_1 + 3y_2 \leq 1, \\ &\quad 2y_1 + y_2 \leq 1. \end{aligned}$$

Here, we have the optimal basis as:

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}, \qquad c_B^\intercal = \begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad \hat{y}^\intercal = c_B^\intercal B^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \end{pmatrix}.$$

Here, we have the inequality that:

$$\begin{cases} \frac{7}{5} = 2 \cdot \frac{2}{5} + 3 \cdot \frac{1}{5} \leq 4, \\ 1 = 1 \cdot \frac{2}{5} + 3 \cdot \frac{1}{5} \leq 1, \\ 1 = 2 \cdot \frac{2}{5} + 1 \cdot \frac{1}{5} \leq 1. \end{cases}$$

Hence, we have the objective function value as $4 \cdot \frac{2}{5} + 3 \cdot \frac{1}{5} = \frac{11}{5}$.
Moreover, we have the slacks as $\left(\frac{13}{5}, 0, 0\right)$, which is positive and corresponds to the top row of the LP tableau. ◊

Remark III.1.12. In fact, the slacks should be the negative of the top row. This is because that we can write the transpose of the slack as:

$$c^\intercal - \hat{y}^\intercal A = c^\intercal - c_B^\intercal B^{-1} A.$$

⌟

Corollary III.1.13. For the LP and its corresponded DP, there are exactly 4 possibilities:

(i) LP and DP are both feasible, they have optimal solutions with objective function values equal to each other.

(ii) LP is unbounded, *i.e.*, the objective function is not bounded below, hence DP is infeasible.

(iii) LP is infeasible and DP has objective function unbounded, *i.e.*, the objective function is not bounded above.

(iv) LP and DP are infeasible.

Example III.1.14. Extreme Case Study.
Consider $A \in \mathbb{R}^{m \times n}$ are all zeros, we have:

- The LP is feasible if and only if $b = 0$, in which there will be two possibilities:
  - It is optimal objective function value if $c \geq 0$.
  - The objective function value is unbounded if $c \ngeq 0$.
- The DP is feasible if and only if $c \geq 0$.
  - It is optimal objective function value if $b = 0$.
  - The objective function value is unbounded if $b \neq 0$.

Therefore, the four cases follows along:

	$b = 0$	$b \neq 0$
$c \geq 0$	(i)	(iii)
$c \ngeq 0$	(ii)	(iv)

Table III.1. Four possibilities in linear program and dual program.

◊

III.2 Small Deviation on Constraints

Now, think of the LP as:

$$\begin{aligned} &\min c^\intercal x, \\ &\text{s.t. } Ax = b, \\ &\quad x \geq 0, \end{aligned}$$

in which we can obtain the final tableau, and consider for the example, the constraints are changed, so suppose now, the new LP' will be:

$$\begin{aligned} &\min c^\intercal x, \\ &\text{s.t. } Ax = b + \Delta b, \\ &\quad x \geq 0. \end{aligned}$$

Therefore, we would have the final tableau as:

1	$0^\intercal$	$-c_N^\intercal + c_B^\intercal B^{-1}N$	$c_B^\intercal B^{-1}(b + \Delta b)$
0	Id	$B^{-1}N$	$B^{-1}(b + \Delta b)$

Table III.2. Final tableau with new data input.

Remark III.2.1. Stability of Small Deviation.
If the original final tableau is nondegenerate, *i.e.*, $B^{-1}b > 0$, then for any Δb small enough, it would still hold for $B^{-1}(b + \Delta b) \geq 0$ and as such, the *basic feasible tableau* is still optimal. ⌟

Now, assuming that Δb is small enough, and we consider the oofv as $\hat{y}^\intercal b$ and becomes:

$$\hat{y}(b + \Delta b) = \hat{y}^\intercal b = \hat{y}^\intercal \Delta b,$$

so we have that:

$$\frac{\partial \text{ oofv}}{\partial b_i} = \hat{y}_i.$$

Definition III.2.2. Complementary Vector.
The two vectors are **complementary vectors** if the nonzero vectors of one vector correspond to the zeros in the other. ⌟

Example III.2.3. Consider the vectors:

$$x = \begin{pmatrix} 63 \\ 0 \\ 0 \\ 9 \\ 0 \\ 2 \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 0 \\ 10 \\ 0 \end{pmatrix},$$

and they are complementary. ◊

Proposition III.2.4. Monotonicity of Complementary Vector.
If $x \geq 0$ and $y \geq 0$ or $x \leq 0$ and $y \leq 0$, x and y are complementary if and only if $x^\intercal y = 0$.

Proof. Trivial, as the sign for the sum of elements of same sign is preserved. □

Theorem III.2.5. Complementary Slackness.
If x is feasible in LP and y is feasible in DP, then x and y are respectively optimal in LP and DP if and only if x is complementary to dual slack of y.

Proof. Suppose that x is feasible in LP, *i.e.*, $Ax = b$ for $x \geq 0$, and y is feasible in DP, *i.e.*, $A^\intercal y \leq c$. Therefore, we have:

$$b^\intercal y = (Ax)^\intercal y = x^\intercal A^\intercal y \leq x^\intercal c = c^\intercal x. \tag{6}$$

Recall from **strong duality** (Theorem III.1.10) and **supervisor principle** (Corollary III.1.8), x and y is optimal if and only if $b^\intercal y = c^\intercal x$, which happens if and only if the equality holds in (6), *i.e.*:

$$x^\intercal (c - A^\intercal y) = 0,$$

which is, equivalently, x is complementary to $c - A^\intercal y$ since $x^\intercal$ is nonnegative and $c - A^\intercal y$ is also nonnegative. □

Alternatively, we can also prove this using the canonical form, *i.e.*, we can similarly argue.

Proof. Suppose x is feasible in LP means $Ax \geq b$ and $x \geq 0$, while y is feasible in DP means that $A^\intercal y \leq c$ and $y \geq 0$, and we can similarly form that:

$$b^\intercal y \leq (Ax)^\intercal y = x^\intercal A^\intercal y \leq x^\intercal c = c^\intercal x.$$

Hence, x and y are respectively optimal in LP and DP if and only if $b^\intercal y = c^\intercal x$, *i.e.*, it is equivalently:

$$(Ax - b)^\intercal y = 0 \qquad \text{and} \qquad x^\intercal (c - A^\intercal y) = 0,$$

hence we have, equivalently, $Ax - b$ is complementary to y and x is complimentary to $c - A^\intercal y$. □

III.3 Basic Tableau for Dual Problem

Now, we shift back to the simplex tableau, consider the LP:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t. } & Ax = b, \qquad \text{where } A \in R^{m\times n} \\ & x \geq 0. \end{aligned}$$

Here, the simplex pre-tableau is:

1	$-c^{\intercal}$	0
0	A	b

Table III.3. Simplex pre-tableau for intial setup.

After we execute row reductions, we have:

1	$0^{\intercal}$	...	...
0	Id	...	$B^{-1}b$

Table III.4. Simplex tableau, but not necessarily feasible.

Remark III.3.1. We can quite normally, without loss of generality, that $B \in \mathbb{R}^{m\times m}$ is invertible, but $B^{-1}b$ may well not be feasible. ⌟

Remark III.3.2. In general, the chances that $B^{-1}b$ being nonnegative is small, which in that case, we can have a basic tableau. ⌟

Here, we consider the primal vector $\tilde{x}$ with basic variables $B^{-1}b$ for non basic variables being zero, and the dual vector $\tilde{y}$, we have $\tilde{y}^{\intercal} = c_B^{\intercal}B^{-1}$.

Consider the optimality checklist, we have now:

- Is $\tilde{x}$ feasible in LP?

$$\text{(i) Is } A\tilde{x} = b? \qquad \text{(ii) Is } \tilde{x} \geq 0?$$

- Is $\tilde{y}$ feasible in DP?

$$\text{(iii) Is } A^{\intercal}\tilde{y} \leq c?$$

- Does $\text{ofv}_{\text{DP}}(\tilde{y}) = \text{ofv}_{\text{LP}}(\tilde{x})$?

$$\text{(iv) Is } c^{\intercal}\tilde{x} = b^{\intercal}\tilde{y}?$$

Notice that (i) is naturally true, as row operations do not change the satisfaction of linear system, and (iv) is also true since:

$$\tilde{y}^{\intercal} = c_B^{\intercal}B^{-1}b = c_B^{\intercal}\tilde{x}_B = c^{\intercal}\tilde{x}.$$

Since we want all conditions on the checklist to satisfy, we consider the remaining two conditions:

(ii) This is precisely when the right hand side of the tableau $B^{-1}b \geq 0$.

(iii) This is equivalently that $-c + A^{\intercal}\tilde{y} \leq 0$, *i.e.*, $-c^{\intercal} + \tilde{y}^{\intercal}A = 0$, so this is when the top row of the basic tableau to be ≤ 0.

	Primal Simplex Algorithm	Dual Simplex Algorithm
(ii)	Start with (ii)	towards (ii)
	↓	↑
(iii)	towards (iii)	Start with (iii)

Remark III.3.3. In the simplex algorithm, we always have (i) and (iv) holding, and we identically have (ii) as being x feasible (right hand side nonnegative) and (iii) as being y feasible (top row nonpositive). ⌟

Now, consider that if we solve the LP to final tableau, and if b changes (significantly, not small deviation since Remark III.2.1) so the right hand side is nonnegative.

Also, we can consider a problem casted with another more condition:

$$\begin{aligned} \min\ & c^\intercal x, \qquad \text{if } c \geq 0, \\ \text{s.t. } & Ax \geq b, \\ & x \geq 0. \end{aligned}$$

This makes perfect sense to use the dual simplex algorithm.

Example III.3.4. Consider the problem:

$$\begin{aligned} \min\ & 4x_1 + x_3 + 5x_3 + 2x_4, \\ \text{s.t. } & -x_1 + 2x_2 + 4x_3 - 2x_4 \geq \frac{5}{2}, \\ & 2x_1 + 3x_2 - x_3 + 5x_4 \geq 0, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Now, we make this this standard form:

$$\begin{aligned} \min\ & 4x_1 + x_3 + 5x_3 + 2x_4, \\ \text{s.t. } & -x_1 + 2x_2 + 4x_3 - 2x_4 - x_5 = \frac{5}{2}, \\ & 2x_1 + 3x_2 - x_3 + 5x_4 - x_6 = 0, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

Then, we want to make all the slack variables to be represented as positive, so we have the tableau as:

1	-4	-1	-5	-2	0	0	0
0	1	-2	-4	2	1	0	$-\frac{5}{2}$
0	-2	-3	1	-5	0	1	-3

Table III.5. The basic tableu (although not feasible) with top row being nonpositive.

Then, we would need a way to pivot, and we will still use the greedy approach. Consider -3 is most negative, so we pivot on the second row. Consider how much we can change before making the top row positive, we have:

$$\begin{aligned} c_1 &= -4 + 2x_1 && \text{when } x_1 > 2,\ c_1 \text{ will go positive.} \\ c_2 &= -1 + 3x_2 && \text{when } x_2 > \tfrac{1}{3},\ c_2 \text{ will go positive.} \\ c_3 &= -5 - x_3 && \text{which will not go positive.} \\ c_4 &= -2 + 5x_4 && \text{when } x_4 > \tfrac{2}{5},\ c_3 \text{ will go positive.} \end{aligned}$$

Therefore, we will be pivoting on the second column, in which we have:

1	$-\frac{10}{3}$	0	$-\frac{16}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	1
0	$\frac{7}{3}$	0	$-\frac{14}{3}$	$\frac{11}{3}$	1	$-\frac{2}{3}$	$-\frac{1}{2}$
0	$\frac{2}{5}$	1	$-\frac{1}{3}$	$\frac{5}{3}$	0	$-\frac{1}{3}$	1

Table III.6. The basic tableu still with top row being nonpositive.

Then, we pivot again greedily, pivoting the first row and the last column which gives us that:

1	$-\frac{9}{2}$	0	-3	-3	$-\frac{1}{2}$	0	$\frac{5}{4}$
0	$-\frac{7}{2}$	0	7	-8	$-\frac{3}{2}$	1	$\frac{3}{4}$
0	$-\frac{1}{2}$	1	2	-1	$-\frac{1}{2}$	0	$\frac{5}{4}$

Table III.7. The feasible basic tableu with top row being nonpositive.

Note that the right hand side entries are nonnegative, so it is feasible, and we have effectively solved the problem. ◊

Remark III.3.5. Again, consider again with the case that a row has only positive entries when that row is the only negative entry on the right hand side, for example:

1	0	0			
0	1	0			+
0	0	1	+	+	-

Table III.8. The infeasible basic tableu in the degeneracy case.

In this case, we have the problem being infeasible.
Similarly, when we have 0 and negative, we will have the stall situation:

1	0	0	0	
0	1	0		
0	0	1	+	-

Table III.9. The infeasible basic tableu in the stall case.

┘

Part 3

Nonlinear Optimization Problem

IV Mathematic Foundations on Vector Space

IV.1 Preliminaries on Linear Algebra

Definition IV.1.1. Given a symmetric matrix $A \in \mathbb{R}^{n \times n}$, it is called:

- **positive definite** (PD) if for all nonzero $d \in \mathbb{R}^n$, $d^\intercal A d > 0$.
- **positive semi-definite** (PSD) if for all nonzero $d \in \mathbb{R}^n$, $d^\intercal A d \geq 0$.
- **negative definite** (ND) if for all nonzero $d \in \mathbb{R}^n$, $d^\intercal A d < 0$.
- **negative semi-definite** (NSD) if for all nonzero $d \in \mathbb{R}^n$, $d^\intercal A d \leq 0$.
- **indefinite** otherwise.

⌟

Example IV.1.2. Consider the matrix $\begin{pmatrix} 3 & 2 & 6 \\ 2 & 1 & -1 \\ 6 & -1 & 5 \end{pmatrix}$, we note that the **Hessian matrix** is:

$$\begin{pmatrix} d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 2 & 1 & -1 \\ 6 & -1 & 5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 3d_1^2 + 2d_1d_2 + 6d_1d_3 + 2d_2d_1 + d_2^2 - d_2d_3 + 6d_3d_1 - d_3d_2 + 5d_3^2.$$

It would be dependent on the sign of the right hand side. ◊

Theorem IV.1.3. For any symmetric $A \in \mathbb{R}^{m \times m}$:

- positive definite if and only if all eigenvalues of A are positive.
- positive semi-definite if and only if all eigenvalues of A are nonnegative.
- negative definite if and only if all eigenvalues of A are negative.
- negative semi-definite if and only if all eigenvalues of A are nonpositive.

Remark IV.1.4. The eigenvalues of the matrix of Example IV.1.2 is approximately 10.11, -3.21, and 2.09, so it is indefinite. ⌟

It turns out that the eigenvalues is a good way to determine if a matrix is definite.

Example IV.1.5. Consider the following matrices:

- $\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$ has eigenvalues $\{7, 2\}$, so it is positive definite.
- $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ has eigenvalues $\{4, 0\}$, so it is positive semi-definite.
- $\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$ has eigenvalues $\{\frac{3}{2}, -\frac{1}{2}\}$, so it is indefinite.

$\diamond$

Proposition IV.1.6. Consider $A \in \mathbb{R}^{n\times n}$ being symmetric, suppose that A is positive semi-definite and invertible, then A is positive definite.

Proof. Since A is invertible, its determinant must be nonzero, and since the determinant is the product of eigenvalues, so A does not have a negative eigenvalue. $\square$

Proposition IV.1.7. If A is invertible, eigenvalues of A^{-1} are reciprocals of the eigenvalues of A.

Proof. Suppose $\xi_1, \cdots, \xi_k$ are eigenvectors of A associated with eigenvalues $\lambda_1, \cdots, \lambda_k$, then we have A^{-1} mapping each eigenvector to $\frac{1}{\lambda_1}, \cdots \frac{1}{\lambda_k}$ multiples of itself as preimage. $\square$

Therefore, we have a direct consequence as follows:

Corollary IV.1.8. Suppose $A \in \mathbb{R}^{m\times m}$ is symmetric, then A is positive definite if and only if A^{-1} is positive definite.

Lemma IV.1.9. Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and $c \in \mathbb{R}$, the eigenvalues of cA are c times each eigenvalues of A, and the eigenvalues of $c\,\mathrm{Id} + A$ are c plus each eigenvalues of A.

Proof. Consider A is a linear operator and x is an eigenvector associated to eigenvalue λ, we have $(cA)x = c(Ax) = c(\lambda x) = (c\lambda)x$ and $(c\,\mathrm{Id} + A)x = c\,\mathrm{Id}\,x + Ax = cx + \lambda x = (c + \lambda)x$. $\square$

IV.2 Preliminaries on Calculus

Then, we will be introducing to our old friend, the Taylor formula, used to find the expansion of sufficiently smooth functions into polynomials.

Theorem IV.2.1. Taylor's Theorem.
Suppose $f : (a, b) \to \mathbb{R}$ is sufficiently smooth up to the k-th derivative (*i.e.*, $f \in \mathcal{C}^k\big((a, b)\big)$) for some

nonnegative k and $\bar{x} \in (a, b)$, then for any $x \in (a, b)$:

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(\bar{x})}{2}(x - \bar{x})^2 + \cdots + \frac{f^{(k)}(\bar{x})}{k!}(x - \bar{x})^k + \underbrace{\frac{f^{(k+1)}(\xi)}{(k+1)!}(x - \bar{x})^{k+1}}_{\text{remainder form}},$$

for some η between x and $\bar{x}$.

Remark IV.2.2. Note that the ξ in this definition is **not** constructive, its existence is guaranteed by the **Cauchy's mean value theorem**. ⌟

Example IV.2.3. Consider that we have $f(0) = 2$, $f'(0) = -1$, and $f''(0) = 1$:

- Use constant approximation, we consider $g(x) = 2$, *i.e.*:

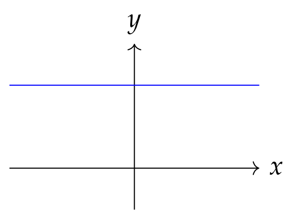


Figure IV.1. 0th order approximation for f about 0.

- Use linear approximation, we consider $g(x) = 2 - x$, *i.e.*:

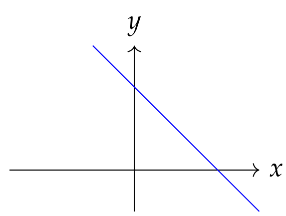


Figure IV.2. 1st order approximation for f about 0.

- Use quadratic approximation, we consider $g(x) = 2 - x + x^2$, *i.e.*:

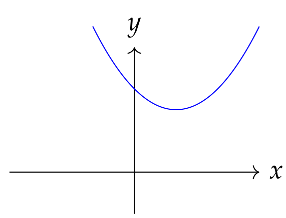


Figure IV.3. 2nd order approximation for f about 0.

◊

Then, we think about the higher dimensional analogue of derivatives, we have:

Definition IV.2.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be sufficiently smooth (*i.e.*, $\mathcal{C}^2(\mathbb{R}^n)$), the **gradient** of f is defined as:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix},$$

and the **Hessian** is defined to be:

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$$

⌟

Example IV.2.5. Consider the function $f(x_1, x_2) = e^{x_1+x_2} + \frac{x_1^2}{2} + \frac{x_2^2}{2} - x_1 - 2x_2$, we have:

$$\nabla f(x_1, x_2) = \begin{pmatrix} e^{x_1+2x_2} + x_1 - 1 \\ 2e^{x_1+2x_2} + x_2 - 2 \end{pmatrix},$$

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} e^{x_1+2x_2} + 1 & 2e^{x_1+x_2} \\ 2e^{x_1+2x_2} & 4e^{x_1+x_2} + 1 \end{pmatrix} = \mathrm{Id} + e^{x_1+x_2} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Moreover, at $x = 0$, we have:

$$\nabla f(0) = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \nabla^2 f(0) = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}.$$

The matrix in the end is $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ with eigenvalues 0 and 5. Recall from Lemma IV.1.9, the eigenvalues of the Hessian are 1 and $5e^{x_1+2x_2} + 1$, which are positive. Hence, the Hessian is positive definite.

Note that Hessian being positive definite means that it is convex, and with a stationary point at 0, this means that we would have a minimum at 0 for the problem.

◊

Furthermore, one can, of course, define higher order derivative of functions taking in vector valued input, namely through **tensors** (or multilinear maps).

Definition IV.2.6. Higher Derivatives.
Given $f : \mathbb{R}^n \to \mathbb{R}$ and some $a \in \mathbb{R}^n$ fixed, we define the d-th derivative as $D^d f(a) : (\mathbb{R}^n)^d \to \mathbb{R}$ in which:

$$\left(D^d f(a)\right)^{i_1, i_2, \cdots, i_d} = \left.\frac{\partial f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_d}}\right|_a.$$

Therefore, consider vectors $v_1, v_2, \cdots, v_d \in \mathbb{R}^n$, we have:

$$D^d f(a)\left[v_1, v_2, \cdots, v_d\right] = \sum_{i_1, i_2, \cdots, i_d = 1}^{n} \left(D^d f(a)\right)^{i_1, i_2, \cdots, i_d} v_1^{i_1} v_2^{i_2} \cdots v_d^{i_d},$$

which can be written as $D^d f(a)[v_1, v_2, \cdots, v_d] = (D^d f(a))^{i_1, i_2, \cdots, i_d} v_1^{i_1} v_2^{i_2} \cdots v_d^{i_d}$ using Einstein summation convention. ⌟

Another common class would be the quadratic functions, commonly written as:

$$f(x_1, x_2) = Ax_1^2 + Bx_2^2 + Cx_1x_2 + Dx_1 + Ex_2 + F,$$

where $A, B, C, D, E, F \in \mathbb{R}$ are constants.

Example IV.2.7. Consider the function:

$$\begin{aligned} f(x_1, x_2) &= 3x_1^2 - 2x_1x_2 + \frac{3}{2}x_2^2 + 8x_1 + 9x_2 - 11 \\ &= \frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 11. \end{aligned}$$

It is not hard to also notice that:

$$\nabla f(x_1, x_2) = \begin{pmatrix} 6x_1 - 2x_2 + 8 \\ -2x_1 + 3x_2 + 9 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x_1, x_2) = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix},$$

and this is not a coincidence. ◊

Proposition IV.2.8. Derivatives of Quadratic Functions.
Consider $A \in \mathbb{R}^{n \times n}$ is symmetric, let $B \in \mathbb{R}^n$ and $c \in \mathbb{R}$. For function $f(x) = \frac{1}{2}x^\top Ax + b^\top x + c$, we have:

$$\nabla f(x) = Ax + b \quad \text{and} \quad \nabla^2 f(x_1, x_2) = A.$$

Proof. Arithmetics. □

Remark IV.2.9. For any $\bar{x} \in \mathbb{R}^n$, we can form the polynomial as:

$$f(x) = \frac{1}{2}(x - \bar{x})^\top A(x - \bar{x}) + b^\top (x - \bar{x}) + c,$$

whose derivatives can be computed as:

$$\nabla f(x) = A(x - \bar{x}) + b \quad \text{and} \quad \nabla^2 f(x) = A.$$

⌟

Then, we can move our attention back to the Taylor polynomial and extend it to the higher dimensions.

Theorem IV.2.10. Taylor's Theorem (Higher Dimension).
Consider $f : \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth and $\bar{x} \in \mathbb{R}^b$. We can approximate f with g as follows:

- Zeroth order approximation at $\bar{x}$: $g(x) = f(\bar{x})$.

- First order approximation at $\overline{x}$: $g(x) = \nabla f(\overline{x})^\intercal (x - \overline{x}) + f(\overline{x})$.
- Second order approximation at $\overline{x}$: $g(x) = \frac{1}{2}(x - \overline{x})^\intercal \nabla^2 f(\overline{x})(x - \overline{x}) + \nabla f(\overline{x})^\intercal (x - \overline{x}) + f(\overline{x})$.

Remark IV.2.11. The *first order approximation* at a point $\overline{x}$ is the tangent hyperplane to the curve across $\overline{x}$, denoted $T_{\overline{x}}\Gamma_f$. ⌟

Remark IV.2.12. Higher Degree Taylor's Theorem.
Likewise, if f is sufficiently smooth, we can approximate f with finner grained Taylor's theorem:

$$g(x) = f(\overline{x}) + Df(\overline{x})^i (x - \overline{x})^i + \frac{1}{2} D^2 f(\overline{x})^{i,j} (x - \overline{x})^{i,j} + \frac{1}{6} D^3 f(\overline{x})^{i,j,k} (x - \overline{x})^{i,j,k} + \cdots .$$ ⌟

Now, we consider the function $f : \mathbb{R}^n \to \mathbb{R}$ being sufficiently smooth, and suppose $\overline{x}, d \in \mathbb{R}^n$. Now, we construct the function:

$$h_d(\alpha) := f(\overline{x} + \alpha d), \qquad \text{where } \alpha \in \mathbb{R}.$$

We have the directional derivatives as:

$$\frac{dh_d}{d\alpha} = \nabla f^\intercal d \qquad \text{and} \qquad \frac{d^2 h_d}{d\alpha^2} = d^\intercal \nabla^2 f d.$$

Remark IV.2.13. Recall from the one-dimensional Taylor polynomial, we have the remainder terms:

$$\begin{aligned} h_d(\alpha) &= h_d(0) + h_d'(\xi)(\alpha - 0), \\ h_d(\alpha) &= h_d(0) + h_d'(0)(\alpha - 0) + \frac{h''(\zeta)}{2}(\alpha - 0)^2. \end{aligned}$$

Now, when we generalize to higher order, we have:

$$\begin{aligned} f(\overline{x} + \alpha d) &= f(\overline{x}) + \nabla f(z)^\intercal \alpha d, \\ f(\overline{x} + \alpha d) &= f(\overline{x}) + \nabla f(\overline{x})^\intercal \alpha d + \frac{d^\intercal \nabla f(\tilde{z}) d}{2} \cdot \alpha^2. \end{aligned}$$

Then with the Taylor's plug-in, we have:

$$\begin{aligned} f(x) &= f(\overline{x}) + \nabla f(z)^\intercal (x - \overline{x}), \\ f(x) &= f(\overline{x}) + \nabla f(z)^\intercal (x - \overline{x}) + \frac{1}{2}(x - \overline{x})^\intercal \nabla^2 f(z)(x - \overline{x}), \end{aligned}$$

which aligns with the Taylor's polynomial. ⌟

Now, recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is smooth enough, given some $\overline{x} \in \mathbb{R}^n$, we have:

$$\begin{aligned} f(x) &\approx f(\overline{x}), \\ f(x) &\approx f(\overline{x}) + \nabla f^\intercal(\overline{x})(x - \overline{x}), \\ f(x) &\approx f(\overline{x}) + \nabla f^\intercal(\overline{x})(x - \overline{x}) + \frac{1}{2}(x - \overline{x})^\intercal \nabla^2 f(\overline{x})(x - \overline{x}). \end{aligned}$$

However, when we have a specific direction $v \in \mathbb{R}^n$, we can have:

$$h(\alpha) = f(\overline{x} + \alpha v) \qquad \text{and} \qquad h'(\alpha) = \nabla f^\mathsf{T} v,$$

noting that we have:

$$\begin{aligned} h(\alpha) &= f(\overline{x} + \alpha v) = f(\overline{x}) + \nabla f^\mathsf{T}(z) v \cdot \alpha, \\ h(\alpha) &= f(\overline{x} + \alpha v) = f(\overline{x}) + \nabla f^\mathsf{T}(\overline{x}) v \cdot \alpha + \frac{1}{2} v^\mathsf{T} \nabla^2 f(z) v \cdot \alpha^2. \end{aligned}$$

Hence, we have:

$$\begin{aligned} f(x) &= f(\overline{x}) + \nabla f^\mathsf{T}(z)(x - \overline{x}), \\ f(x) &= f(\overline{x}) + \nabla f^\mathsf{T}(\overline{x})(x - \overline{x}) + \frac{1}{2}(x - \overline{x})^\mathsf{T} \nabla^2 f(z)(x - \overline{x}), \end{aligned}$$

where we have $z \in \mathbb{R}^n$ to be dependent on the input.

V Unconstrained Nonlinear Optimization

V.1 Gradient Direction and Criterion for Minimum

When we move towards nonlinear optimization problem, there will not be as good of the properties as linear optimization problem.

Definition V.1.1. Linear Descent Direction.
Let $f : \mathbb{R}^n \to \mathbb{R}$ being continuously differentiable, suppose $\overline{x} \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$ is nonzero. We consider d as a (linear) descent direction precisely when:

$$\nabla f(\overline{x})^\intercal d < 0.$$

⌟

This implies that $f(\overline{x} + \alpha d) < f(\overline{x})$ for all $\alpha > 0$ to be small enough.

Remark V.1.2. Technically, one can restrict d such that $\|d\| = 1$, but for notational purposes, we will not restrict the condition. ⌟

While we consider the steepest descent direction, one should notice that technically, we need to divide $\|d\|$ eventually to prevent $\|d\|$ exploding as inner product with $\nabla f(\overline{x})$ is a **linear functional**.

Proposition V.1.3. The steepest descent direction is $-\nabla f(\overline{x})$.

Proof. Without loss of generality, suppose that $\|d\| = 1$, then the steepest descent (as well as increment) is when $d \parallel \nabla f(\overline{x})$. Formally speaking, we can use the Cauchy-Schwartz inequality that:

$$|d^\intercal \nabla f(\overline{x})| \leq \|d\| \cdot \|\nabla f(\overline{x})\| = \|\nabla f(\overline{x})\| = |\nabla f(\overline{x})^\intercal \nabla f(\overline{x})|.$$

Hence, for any unit vector d, $|d^\intercal \nabla f(\overline{x})| \leq |\nabla f(\overline{x})^\intercal \nabla f(\overline{x})|$.
Without the absolute sign, we know that:

$$-\nabla f(\overline{x})^\intercal \nabla f(\overline{x}) \leq d^\intercal \nabla f(\overline{x}) \leq \nabla f(\overline{x})^\intercal \nabla f(\overline{x}),$$

so we have the steepest descent (and increment) directions shown. □

Then, we consider the 1$^{\text{st}}$ order necessary optimality condition.

Theorem V.1.4. 1$^{\text{st}}$ Order Necessary Optimality Condition.
Suppose $S \subset \mathbb{R}^n$ is open, $\overline{x} \in S$, and $f : S \to \mathbb{R}$ is continuously differentiable. If $\overline{x}$ is a local minimum, then $\nabla f(\overline{x}) = 0$.

Proof. If $\nabla f(\overline{x}) \neq 0$, then there exits a descent direction (*i.e.*, $-\nabla f(x)$) such that $\overline{x}$ is not optimal. □

Following that, we consider the 2^{nd} order necessary optimality condition.

Theorem V.1.5. 2^{nd} Order Necessary Optimality Condition.
Suppose $S \subset \mathbb{R}^n$ is open, $\overline{x} \in S$, and $f : S \to \mathbb{R}$ is twice continuously differentiable. If $\overline{x}$ is a local minimum, then $\nabla^2 f(\overline{x})$ is positive semidefinite.

Proof. Suppose $\overline{x}$ is a local minimum, but $\nabla^2 f(\overline{x})$ is not positive semidefinite, then there exists some (nonzero) $d \in \mathbb{R}^n$ such that $d^\intercal \nabla^2 f(\overline{x}) d < 0$.
Now, consider **Taylor's theorem** that:

$$f(\overline{x} + \alpha d) = f(\overline{x}) + \nabla f^\intercal(\overline{x}) d \cdot \alpha + \frac{1}{2} d^\intercal \nabla^2 f(z) d \cdot \alpha^2.$$

Recall from the previous theorem, we have $f^\intercal(\overline{x}) = 0$, and hence for $\alpha > 0$ sufficiently small, we have $d^\intercal \nabla^2 f(z) d \cdot \alpha^2$ being negative by continuity of $\nabla^2 f$, which contradicts that we have local minimum at $\overline{x}$. □

Now, we will provide a 2^{nd} order sufficient optimality condition.

Theorem V.1.6. 2^{nd} Order Sufficient Optimality Condition.
Suppose $S \subset \mathbb{R}^n$ is open, $\overline{x} \in S$, and $f : S \to \mathbb{R}$ is twice continuously differentiable. If $\nabla f(\overline{x}) = 0$ and $\nabla^2 f(\overline{x})$ is positive definite, then $\overline{x}$ is a strict local minimum.

Proof. Without loss of generality, we suppose $d \in \mathbb{R}^n$ to be of length 1 and $\alpha > 0$ to be small enough. Consider **Taylor's theorem** again, we have:

$$f(\overline{x} + \alpha d) = f(\overline{x}) + \nabla f^\intercal(\overline{x}) d \cdot \alpha + \frac{1}{2} d^\intercal \nabla^2 f(z) d \cdot \alpha^2.$$

By assumption, we have $f^\intercal(\overline{x}) = 0$ and ${}^\intercal \nabla^2 f(z) d > 0$ by positive definiteness and by continuity of eigenvalues of $\nabla^2 f(z)$. □

Remark V.1.7. When we lift the condition of the previous theorem to positive semidefinite, this is not true, even if we lift the strict local minimum local minimum, this is not true, due to the existence of *saddle points*. ⌟

Example V.1.8. Saddle Point in Local Minimum.
We consider the function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^3$, we have $f'(0) = 0$ and $f''(0) = 0$, so we have $\nabla f(0) = \begin{pmatrix} 0 \end{pmatrix}$ and $\nabla^2 f(0) = \begin{pmatrix} 0 \end{pmatrix}$, but this is not a local minimum since we have $f(x) > 0$ for all $x > 0$. ♢

V.2 Convexity

Recall the definition of a convex set, $S \subset \mathbb{R}^n$ is convex precisely when for all $x, y \in S$, for all $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in S$.

We can extend this definition to functions as well:

Definition V.2.1. Convex Functions.
Suppose $S \in \mathbb{R}^n$ is a convex set, $f : S \to \mathbb{R}^n$ is a convex function precisely when for all $x, y \in S$ and $\lambda \in [0,1]$:

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

f is concave if $-f$ is convex. ⌟

Here, we can give an example of convex function in $\mathbb{R} \to \mathbb{R}$:

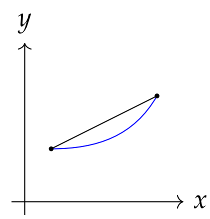


Figure V.1. A Convex function in $\mathbb{R} \to \mathbb{R}$.

Lemma V.2.2. f is concave if and only if for all $x, y \in S$ and $\lambda \in [0,1]$:

$$f\big(\lambda x + (1-\lambda)y\big) \geq \lambda f(x) + (1-\lambda) f(y).$$

Proof. By definition. □

Proposition V.2.3. A function is both convex and concave if and only if it is affine.

Proof. ($\Longrightarrow$:) Suppose f is convex and concave, this implies that:

$$f\big(\lambda x + (1-\lambda)y\big) = \lambda f(x) + (1-\lambda) f(y).$$

Here, we have that for all $a, b \in \left(0, \frac{1}{2}\right]$, there exists some c such that $a + b + c = 1$, and so we have:

$$f(ax + by) = f(0 \cdot c + ax + by) = cf(0) + af(x) + bf(y),$$

hence, we have f being locally affine, which implies that f is affine.
($\Longleftarrow$:) Suppose f is affine, then we can write:

$$f(x) = a^{\mathsf{T}} x + b,$$

hence, for any $\lambda \in [0,1]$ and $x, y \in S$, we have that:

$$f\big(\lambda x + (1-\lambda)y\big) = a^\intercal\big(\lambda x + (1-\lambda)y\big) + b = \lambda a^\intercal x + \lambda b + (1-\lambda)a^\intercal y + (1-\lambda)b = \lambda f(x) + (1-\lambda)f(y),$$

which proves the other direction. □

Definition V.2.4. Strictly Convex.
Suppose $S \in \mathbb{R}^n$ is a convex set, $f : S \to \mathbb{R}^n$ is a strict convex function precisely when for all $x, y \in S$ and $\lambda \in [0,1]$:

$$f\big(\lambda x + (1-\lambda)y\big) < \lambda f(x) + (1-\lambda)f(y).$$

⌟

Then, we want to characterize convex functions with convex sets.

Definition V.2.5. Epigraph.
Suppose $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$ is epigraph of f is:

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x \in S, y \in \mathbb{R} \text{ such that } y \geq f(x)\right\} \subset \mathbb{R}^{n+1}.$$

⌟

Proposition V.2.6. Suppose $S \subset \mathbb{R}^n$ is nonempty, convex set, then $f : S \to \mathbb{R}$ is a convex function if and only if epigraph of f is a convex set.

Proof. ($\Longrightarrow$:) Suppose f is convex. Let $(x_1, t_1), (x_2, t_2)$ in the epigraph of f. Hence, we know that $x_1, x_2 \in S$ while also $t_1 \geq f(x_1)$ and $t_2 \geq f(x_2)$.
Now, let $\lambda \in (0,1)$ be arbitrary where we define $(x, t) = \lambda(x_1, t_1) + (1-\lambda)(x_2, t_2)$, we want to show that this point is in the epigraph. Consider consider that f is convex, so:

$$f(x) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \leq \lambda t_1 + (1-\lambda)t_2 = t,$$

so we know that $(x, t) = \lambda(x_1, t_1) + (1-\lambda)(x_2, t_2)$ is in the epigraph.
($\Longleftarrow$:) Suppose f is not convex, there exists some $x_1, x_2 \in S$ and some $\lambda \in (0,1)$ such that:

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) > \lambda f(x_1) + (1-\lambda)f(x_2).$$

We know that $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ are in the epigraph, but:

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) > \lambda f(x_1) + (1-\lambda)f(x_2) = \lambda t_1 + (1-\lambda)t_2,$$

and thus the epigraph of f is not convex. □

Theorem V.2.7. Support Theorem.
Let $S \subset \mathbb{R}^n$ be a convex set, and let $\overline{x}$ be a boundary point of S, then there exists a hyperplane containing

$\overline{x}$ such that S is contain in associated half plane.

Proof. Without loss of generality, we consider $S \neq \mathbb{R}^n$ and $S \neq \emptyset$ since there would not be a boundary set in these cases. Hence, we can assume there exists some $y \in \mathbb{R}^n \setminus (S)_{-1}$ where $(S)_{-1} := \{x \in \mathbb{R}^n : \|x - y\| < 1 \text{ for all } y \in S\}$. Therefore, we can consider the distance:

$$d := \inf_{x \in S} \|x - y\|,$$

note that if we consider $\overline{S}$, all Cauchy sequence converges, so we guaranteed with a unique $\overline{x} \in \partial S$ such that $\|y - \overline{x}\| = d$.

Here, if there exists two $\overline{x} \neq \overline{x}' \in S$ such that $\|y - \overline{x}\| = \|y - \overline{x}'\| = d$, then by convexity (closure of convex set is still convex), we must have some $\tilde{x}$ in between $\overline{x}$ and $\overline{x}'$ exhibiting a shorter distance.

Then, we can create the linear parametrization $\gamma : [0, 1] \to \mathbb{R}^n$ as follows:

$$\gamma(t) = \overline{x} + t(y - \overline{x}),$$

so we have γ as a constant line parametrization of the line segment from $\overline{x}$ to y.
Moreover, we know that $\gamma(t) \notin S$ for all $t > 0$, as otherwise, we would have picked some $\overline{x}$ with some $\gamma(t)$ where $t > 0$, hence we know that the line segment from $\overline{x}$ to y is not in S.
Therefore, we can consider the vector $a = \frac{y - \overline{x}}{\|y - \overline{x}\|}$, which is guaranteed to be nonzero, and we consider the half space:

$$H := \{x \in \mathbb{R}^n : a^\intercal (x - \overline{x}) \leq 0\}.$$

Now, we are left to show that $S \subset H$. For the sake of contradiction, suppose that there exists some $s \in S$ such that $a^\intercal (s - \overline{x}) > 0$, we form the second parametrization $\eta : [0, 1] \to \mathbb{R}^n$ by:

$$\eta(t) = \overline{x} + t(s - \overline{x}),$$

so we now have η as a constant line parametrization of the line segment from $\overline{x}$ to s, here, we can take the inner produce as an affine functional:

$$f(x) = \langle a, x - \overline{x} \rangle = a^\intercal (x - \overline{x}),$$

so we have:

$$f\big(\eta(t)\big) = a^\intercal (\overline{x} + t(s - \overline{x}) - \overline{x}) = (1 - t) a^\intercal (\overline{x} - \overline{x}) + t a^\intercal (s - \overline{x}) = t f(s) > 0 \qquad \text{for } t > 0.$$

Then, we consider the square distance that:

$$\begin{aligned} \|y - \eta(t)\| &= \|\overline{x} + a - \eta(t)\|^2 = \|a + t\overline{x} - ts\|^2 \\ &= \|a\|^2 + 2ta^\intercal (\overline{x} - s) + t^2 \|\overline{x} - s\|^2 = \|a\|^2 - 2tf(s) + t^2 \|\overline{x} - s\|^2. \end{aligned}$$

Note that $\|\overline{x} - s\|^2$ and $f(s)$ are finite and fixed, so we know that there exists some $0 < t < 1$ such that $\|y - \eta(t)\|^2 \leq \|a\|^2$, which implies that $\overline{x}$ is not obtaining the minimum distance, which is a contradiction, hence showing that $S \subset H$, and we complete the proof. $\square$

Proposition V.2.8. Let $S \subset \mathbb{R}^n$ be open and convex, and let $f : S \to \mathbb{R}$ be continuously differentiable. f is a convex function if and only if for all $x, \overline{x} \in S$:

$$f(x) \geq f(\overline{x}) + \nabla f(\overline{x})^\intercal (x - \overline{x}).$$

What this theorem saying is that $f(\overline{x}) + \nabla f(\overline{x})^\intercal (x - \overline{x})$ is the only tangent hyperplane containing the epigraph.

Proof. ($\Longrightarrow$:) Suppose f is convex, then the epigraph of f is convex, hence by the proof **Support Theorem** (Theorem V.2.7), we can construct $a = \nabla f(\overline{x})$ (and we can construct $y \in a^\perp \cap S^c$), and we have $a^\intercal (x - \overline{x}) \leq 0$, so the inequality holds.
($\Longleftarrow$:) When the inequality holds, at each point $(\overline{x}, f(\overline{x}))$, it is bounded below by a hyperplane, which implies that if the epigraph is not convex, in which there exists some $(x, f(x))$ and $(x', f(x'))$ in the epigraph, but (without loss of generality) for some $y = \frac{x+x'}{2}$ that $(y, \frac{f(x)+f(x')}{2})$ not in the epigraph, this implies that the construction would not have $(y, f(y))$ bounded below by the tangent plane, which is a contradiction. ☐

Corollary V.2.9. f is strictly convex function if and only if for all $x, \overline{x} \in S$ such that $x \neq \overline{x}$:

$$f(x) > f(\overline{x}) + \nabla f(\overline{x})^\intercal (x - \overline{x}).$$

Proof. Directly from the previous theorem, replacing $\geq$ to $>$ by strictly convex condition. ☐

Proposition V.2.10. Suppose $S \subset \mathbb{R}^n$ is open and convex. Let $f : S \to \mathbb{R}$ be twice continuously differentiable, then f is a convex function if and only if for all $x \in S$, $\nabla^2 f(x)$ is positive semidefinite.

Proof. ($\Longleftarrow$:) Suppose for all $\overline{x} \in S$, $\nabla^2 f(\overline{x})$ is positive semidefinite.
Now, consider **Taylor's theorem** that:

$$f(x) = f(\overline{x}) + \nabla f^\intercal (x - \overline{x}) + \frac{1}{2}(x - \overline{x})^\intercal \nabla^2 f(z)(x - \overline{x}) \qquad \text{for some } z \text{ between } x \text{ and } \overline{x}.$$

By assumption of convexity for S, $z \in S$, and thus $\nabla^2 f(z)$ is positive semidefinite, thus:

$$f(x) \geq f(\overline{x}) + \nabla f^\intercal (\overline{x})(x - \overline{x}) \qquad \text{for all } x, \overline{x} \in S,$$

and thus f is convex by Proposition V.2.8.
($\Longrightarrow$:) Conversely, suppose that for some $\overline{x} \in S$, $\nabla^2 f(\overline{x})$ is not positive semidefinite. This means that there exists some $d \in \mathbb{R}^n$ such that $d^\intercal \nabla^2 f(\overline{x}) d < 0$.
By the continuity of $\nabla^2 f$, there exists a neighborhood around $\overline{x}$ such that:

$$d^\intercal \nabla^2 f(z) d < 0 \qquad \text{for all } z \text{ in the neighborhood.}$$

Let $x = \overline{x} + \alpha d$ for $\alpha > 0$ to be small enough such that x is in the neighborhood.
Again, consider **Taylor's theorem** that:

$$f(x) = f(\overline{x}) + \nabla f^{\intercal}(x - \overline{x}) + \frac{1}{2} d^{\intercal} \nabla^2 f(z)(x - \overline{x}) \qquad \text{for some } z \text{ between } x \text{ and } \overline{x}.$$

Note that we have established that $\frac{\alpha^2}{2} d^{\intercal} \nabla^2 f(z) d < 0$, so we have that:

$$f(x) < f(\overline{x}) + \nabla f^{\intercal}(\overline{x})(x - \overline{x}),$$

so f is not convex, which completes this proof. □

Corollary V.2.11. Let $S \subset \mathbb{R}^n$ be open and convex set, and $f : S \to \mathbb{R}$ be twice continuously differentiable.
If for all $\overline{x} \in S$, $\nabla^2 f(\overline{x})$ is positive definite, then f is strictly convex.

Proof. For all $\overline{x} \in S$ and $x \in S$ in which $\overline{x} \neq x$, by Taylor's theorem, we have:

$$f(x) = f(\overline{x}) + \nabla f^{\intercal}(x - \overline{x}) + \frac{1}{2} d^{\intercal} \nabla^2 f(z)(x - \overline{x}) \qquad \text{for some } z \text{ between } x \text{ and } \overline{x},$$

and by convexity of the set, $z \in S$, and so $\nabla^2 f(z)$ is positive definite, hence:

$$f(x) > f(\overline{x}) + \nabla f^{\intercal}(\overline{x})(x - \overline{x}),$$

hence f is strictly convex by Corollary V.2.9. □

However, the converse direction does not hold for the above corollary.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$, it is not hard to show that f is strictly convex.
However, we have $f''(x) = 12x^2$, so $f''(0) = 0$, which is not positive definite.

Theorem V.2.12. Suppose $S \subset \mathbb{R}^n$ is open and convex set, and $f : S \to \mathbb{R}$ is continuously differentiable and is a convex function. Consider $\overline{x} \in S$. The following are equivalent:

(i) $\overline{x}$ is a global minimizer.

(ii) $\overline{x}$ is a local minimizer.

(iii) $\nabla f(\overline{x}) = 0$.

Proof. (i) $\Longrightarrow$ (ii): Trivial, by definition.
(ii) $\Longrightarrow$ (iii): By Theorem V.1.4.
(iii) $\Longrightarrow$ (i): If $\nabla f(\overline{x}) = 0$, by convexity in Proposition V.2.8, we that for all $x \in S$:

$$f(x) \geq f(\overline{x}) + \nabla f^{\intercal}(\overline{x})(x - \overline{x}).$$

Thus, $\overline{x}$ is a global minimum. □

Note that if f is strictly convex function, such global minimum is strict.

Example V.2.13. Consider $f(x_1, x_2) = e^{x_1+2x_2} + \frac{x_1^2}{2} + \frac{x_2^2}{2} - x_1 - 2x_2$, the Hessian has eigenvalues of 1 and $1 + 5e^{x_2+2x_2}$, which are both positive, and we note that $\nabla f(0) = 0$, which implies that this is a strict global minimum. ♢

V.3 Newton's Method

The Newton's method is an unconstrained optimization algorithm.

For the algorithm, we suppose $f : \mathbb{R}^n \to \mathbb{R}$ which is twice continuously differentiable. We are seeking $\overline{x} \in \mathbb{R}^n$ such that $\nabla f(\overline{x}) = 0$ (*i.e.*, is a stationary point).

Algorithm V.3.1. Newton's Method.

We start by picking $\overline{x} \in \mathbb{R}^n$.
for $i \leftarrow 1, 2, 3, \cdots$ **do**
  Let $x^{(i+1)} \leftarrow x^{(i)} - \left[\nabla^2 f\left(x^{(i)}\right)\right]^{-1} \nabla f\left(x^{(i)}\right)$.
  if $\nabla f\left(x^{(i+1)}\right) \approx 0$ **then**
    Break out of the loop
  end if
end for

Example V.3.2. Consider the problem:

$$\min_{\text{s.t. } x>0} x^3 - \log x.$$

For $f(x) = x^3 - \log x$, we have:

$$f'(x) = 3x^2 - \frac{1}{x} \qquad \text{and} \qquad f''(x) = 6x + \frac{1}{x^2}.$$

Note that we set the first derivative to be 0, we would have:

$$0 = f'(\overline{x}) = 3\overline{x}^2 - \frac{1}{\overline{x}} \qquad \text{which is } \overline{x} = \frac{1}{\sqrt[3]{3}}.$$

Here, we also observe that $f''(x) > 0$ for all $x > 0$, hence f is *strictly* convex, which implies that $\frac{1}{\sqrt[3]{3}}$ is a strict global minimum.
It seems like we have a result, but have no numerical way of finding a decimal number of this, but we can now apply **Newton's method**.
For Newton Step, consider:

$$-\frac{f'(x)}{f''(x)} = -\frac{3x^2 - \frac{1}{x}}{6x + \frac{1}{x^2}} = \frac{x(1 - 3x^3)}{6x^3 + 1}.$$

Hence, we have the iteration formula formatted as:

$$x^{(i+1)} = x + \frac{x(1-3x^2)}{6x^3+1}.$$

Therefore, we can compute a few intervals:

$$\begin{aligned}
x^{(0)} &= 1, \\
x^{(1)} &= 0.7142\,8571\cdots, \\
x^{(2)} &= 0.6933\,7341\cdots, \\
x^{(3)} &= 0.6933\,6127\,4350\,64\cdots.
\end{aligned}$$

This is notable, since we have $\frac{1}{\sqrt[3]{3}} = 0.69336127435063\cdots$.
Even if we find a smaller point, we have:

$$\begin{aligned}
x^{(0)} &= 0.1, \\
x^{(1)} &= 0.1991\cdots, \\
x^{(2)} &= 0.3847\cdots, \\
x^{(3)} &= 0.6224\cdots, \\
x^{(4)} &= 0.6927\cdots, \\
x^{(5)} &= 0.6933\,6127\,408\cdots.
\end{aligned}$$

Even if we start with $x^{(0)} = 100$, we have $x^{(10)} = 0.6933\,6127\,4330\,63\cdots$. ◊

Remark V.3.3. Newton's method guarantees quadratic convergence ⌟

The above is really a toy example, since we note that Newton's method is capable of solving for higher dimensional maps.

Example V.3.4. Consider the function $f(x_1, x_2) = e^{x_1+2x_2} + \frac{x_1^2}{2} + \frac{x_2^2}{2} - x_1 - 2x_2$, we have:

$$\nabla f(x_1, x_2) = \begin{pmatrix} e^{x_1+2x_2} + x_1 - 1 \\ 2e^{x_1+2x_2} + x_2 - 2 \end{pmatrix} \qquad \text{and} \qquad \nabla^2 f(x_1, x_2) = \mathrm{Id} + e^{x_1+x_2}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

It is not hard to observe that the eigenvalues for $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ are 5 and 0, so the Hessian matrix is positive definite.
Also, note that $\nabla f(0) = 0$, so we have the **Newton's** step as:

$$x^{(i+1)} = x^{(i)} - \left[\nabla^2 f\left(x^{(i)}\right)\right]^{-1} \nabla f\left(x^{(i)}\right).$$

Hence, we would have that:

$$x^{(0)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad x^{(1)} = \begin{pmatrix} 0.068 \\ 0.137 \end{pmatrix}, \qquad x^{(2)} = \begin{pmatrix} 0.009 \\ 0.018 \end{pmatrix}, \qquad \cdots$$

Note that then we have $x^{(3)} \lessapprox 10^{-3}$, $x^{(4)} \lessapprox 10^{-6}$, and $x^{(5)} \lessapprox 10^{-12}$ going further. ◊

Remark V.3.5. Idea for Newton's Method.
Given $f : \mathbb{R}^n \to \mathbb{R}$ being twice continuously differentiable, consider the current $\overline{x} \in \mathbb{R}^n$ and we are seeking $x \in \mathbb{R}^n$ such that $\nabla f(x) = 0$.
We approximate f with second order Taylor polynomial around $\overline{x}$, so we have:

$$f(x) \approx g(x) = f(\overline{x}) + \nabla f^{\mathsf{T}}(\overline{x})(x - \overline{x}) + \frac{1}{2}(x - \overline{x})^{\mathsf{T}} \nabla^2 f(\overline{x})(x - \overline{x}).$$

Hence, we have the gradient of g as:

$$\nabla g(x) = \left[\nabla^2 f(\overline{x})\right](x - \overline{x}) + \nabla f(\overline{x}),$$

and the next iteration of x will be such that $\nabla g(x) = 0$.
When solving for x, we have:

$$\begin{aligned}
\left[\nabla^2 f(\overline{x})\right](x - \overline{x}) + \nabla f(\overline{x}) = 0,& \\
\left[\nabla^2 f(\overline{x})\right](\overline{x} - x) = -\nabla f(\overline{x}),& \\
x - \overline{x} = -\left[\nabla^2 f(\overline{x})\right]^{-1} \nabla f(\overline{x}).&
\end{aligned}$$

In fact, in most computational methods, we will just end with $\Delta x = x - \overline{x}$ and use something like Gaussian elimination to avoid numerical instability. ⌟

Remark V.3.6. If the function is quadratic (including linear or constant function), Newton's method converges in one step.
Otherwise, we can at most obtain a convergent approximation with the algorithm. ⌟

Note that for Newton's method, we have when at $\overline{x} \in \mathbb{R}^n$, we have our new x defined as:

$$x_{\text{new}} = \overline{x} \underbrace{- \left[\nabla^2 f(\overline{x})\right]^{-1} \nabla f(\overline{x})}_{\text{Newton's direction}}.$$

Proposition V.3.7. If $\nabla f(\overline{x}) \neq 0$ and $\nabla^2 f(\overline{x})$ is positive definite, then the Newton's direction is a descent direction.

Proof. Here, we have that:

$$\nabla f^{\mathsf{T}}(\overline{x})\left(\left[-\nabla^2 f(\overline{x})\right]^{-1} \nabla f(\overline{x})\right) = -\nabla f^{\mathsf{T}}(\overline{x})\left[\nabla^2 f(\overline{x})\right]^{-1} \nabla f(\overline{x}) > 0,$$

since $\nabla^2 f(\overline{x})$ is positive definition, so does $\left[\nabla^2 f(\overline{x})\right]^{-1}$. □

Then, we return back to the steepest descent:

Algorithm V.3.8. Method of Steepest Descent.

Given $f : \mathbb{R}^n \to \mathbb{R}$ being continuously differentiable, and we want to minimize f.
Start with $x^{(0)} \in \mathbb{R}^n$ being arbitrary.
for $k \leftarrow 0, 1, 2, \cdots$ **do**
  Let $\alpha^* \leftarrow \arg\min_{\alpha>0} f\big(x^{(k)} - \alpha \nabla f(x^{(k)})\big)$, so α^* is optimal.
  Let $x^{(k+1)} \leftarrow x^{(k)} - \alpha^* \nabla f(x^{(k)})$.
  if $\nabla f\big(x^{(k)}\big) \approx 0$ **then**
    Break out the for loop
  end if
end for

Here, we consider an example in which α^* has closed form computation.

Example V.3.9. Consider $f(x_1, x_2) = (1 + x_1 - x_2)^3 + x_1^4 + x_2^4$, we have:

$$\nabla f(x_1, x_2) = \begin{pmatrix} 3(1 + x_1 - x_2)^2 + 4x_1^3 \\ -3(1 + x_1 - x_2)^2 + 4x_2^3 \end{pmatrix}.$$

Here, we give initial point $x^{(0)} = \binom{0}{0}$, we want to find α^* to solve:

$$\min_{\alpha>0} f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} 3 \\ -3 \end{pmatrix}\right) = \min_{\alpha>0} (1 - 6\alpha)^3 + 162\alpha^4.$$

With some arithmetics, we are minimizing $162\alpha^4 - 216\alpha^3 + 108\alpha^2 - 18\alpha + 1$ with derivative set to 0 as:

$$648\alpha^3 - 648\alpha + 216\alpha - 18 = 0.$$

Technically, with this degree 3 polynomial, there is a **closed form** solution for the roots, *despite it is rather complicated*.
For this problem, the only real root is $\alpha^* = 0.1233\ 4649\ 1684\cdots$, and it must be the global steepest direction for the 1-d problem. Hence, we have:

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \alpha^* \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -0.3700\ 3947\ 5052\cdots \\ 0.3700\ 3947\ 5052\cdots \end{pmatrix}.$$

Notice that we can continue with this step, and we still have degree 3 polynomial, so technically, it is still solvable. ♢

Remark V.3.10. When there could be multiple critical values, it might indicated multiple local minimums, but one can just evaluate the function at all the local minimums to find the optimal α^*. ⌟

One thing that one shall notice is that the trajectory of the optimization vectors will have the next step having *exactly* orthogonal direction from the previous direction, which can be illustrated as follows:

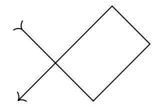


Figure V.2. The directions are all orthogonal.

Here, we have:

$$g(\alpha) = f\big(x^{(k)} - \alpha \nabla f(x^{(k)})\big),$$

where we have α^* is the minimizer for $g(\alpha)$ over $\alpha > 0$, and hence we have:

$$x^{(k+1)} = x^{(k)} - \alpha^* \nabla f(x^{(k)}),$$

thus, we have $0 = g'(\alpha^*) = \nabla f^\intercal(x^{(k+1)})$.

V.4 Theorems of the Alternative

Here, we will explore a few theorems of the alternative that relates to the backgrounds of unconstrained problems.

Theorem V.4.1. Farkas Theorem.
Given a matrix $A \in \mathbb{R}^{m\times n}$ with $b \in \mathbb{R}^m$. Then either:

(i) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$, or

(ii) There exists $y \in \mathbb{R}^m$ such that $b^\intercal y > 0$ and $A^\intercal y \leq 0$,

but not both (exclusive or).

Proof. Consider the following linear program subject to:

$$\min 0^\intercal x, \qquad \text{s.t. } Ax = b, \qquad x \geq 0.$$

Its dual program would be:

$$\max b^\intercal y, \qquad \text{s.t. } A^\intercal y \leq 0, \qquad x \geq 0.$$

If (i) is true, then there exists a feasible x in LP, and the objective function value is 0. By weak duality, DP has no feasible y with an objective function value greater than 0, *i.e.*, (ii) is false.
Conversely, note that DP has a feasible $y = 0$. If (ii) is false, then $y = 0$ is optimal in DP. By strong duality, LP has an optimal solution, which is feasible, hence LP is feasible, and condition (i) is true. □

Then, we consider a special case for the Farkas theorem.

Theorem V.4.2. Gordon's Theorem Given $A \in \mathbb{R}^{m\times n}$. Either:

(i) There exists $x \in \mathbb{R}^n$ such that $Ax = 0$, $x \geq 0$, and x being nonzero, or

(ii) there exists $y \in \mathbb{R}$ such that $A^\intercal y < 0$,

but not both (exclusive or).

Proof. Note that (ii) is true if and only if there exists $y \in \mathbb{R}^m$, $\delta > 0$ such that $A^\intercal y + \delta\vec{1} \leq \vec{0}$, *i.e.*, there exists $\binom{y}{\delta} \in \mathbb{R}^{m+1}$ such that $\binom{\vec{0}}{1}\binom{y}{\delta} > 0$, thence:

$$\begin{pmatrix} A^\intercal \vec{1} \end{pmatrix} \begin{pmatrix} y \\ \delta \end{pmatrix} \leq 0.$$

Hence, we can let $b \leftarrow e_{m+1}$, $y \leftarrow \binom{y}{\delta}$, and $\tilde{A}^\intercal \leftarrow \begin{pmatrix} A^\intercal & \vec{1} \end{pmatrix}$. By Farkas (Theorem V.4.1), this is if and only if there does not exist $x \in \mathbb{R}^n$ such that $\tilde{A}x = b$, *i.e.*, there does not exist $x \in \mathbb{R}^n$ such that $Ax = 0$, the sum of components of x is 1, and $x \geq 0$, *i.e.*, there does not exists $x \in \mathbb{R}^n$ such that $Ax = 0$, x being nonzero, and $x \geq 0$. □

Remark V.4.3. Note that for the last equivalence, the forward is direct, for the converse direction, we can use a scaling with $\alpha > 0$ as the inverse of the sum of the current sum of components. ⌟

VI Constrained Optimization Problem

VI.1 Towards Karush-Kuhn-Tucker Condition

From here, we will start to investigate the constraint problems, *i.e.*, the gradient and Hessian will not be sufficient to conclude optimality.

Hence, we gradually move towards the Karush-Kuhn-Tucker (KKT) Condition for constrained problems.

First, we will be introducing the form of a general unconstrained problem.

Definition VI.1.1. Constrained Problem.
A constrained problem can be formatted as:

$$\begin{aligned} \min\ & f(x), \\ \text{s.t. } & g_1(x) \leq 0, \\ & g_2(x) \leq 0, \\ & \vdots \\ & g_m(x) \leq 0, \end{aligned} \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$.

In particular, we can define $g : \mathbb{R}^n \to \mathbb{R}^n$ by $g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{pmatrix}$, hence we can rewrite the problem as:

$$\begin{aligned} \min\ & f(x), \\ \text{s.t. } & g(x) \leq 0. \end{aligned} \tag{P}$$

⌟

Example VI.1.2. Given a constrained program, *e.g.*, $x_2 e^{x_1} = x_2 + 6$, we have $x_2 e^{x_1} - x_2 - 6 = 0$ and:

$$\begin{cases} x_2 e^{x_1} - x_2 - 6 \leq 0 \\ x_2 e^{x_1} - x_2 - 6 \geq 0 \end{cases}$$

which we now have:

$$\begin{cases} x_2 e^{x_1} - x_2 - 6 \leq 0, \\ -x_2 e^{x_1} + x_2 + 6 \leq 0. \end{cases}$$

◊

It is not too abrupt to assume that f and g to be continuous differentiable, so we can take derivative (or gradients, equivalently) for these functions.

Alas, we will be assuming differentiability for the rest of the chapter.

Definition VI.1.3. Active Constraints.
Consider $x \in \mathbb{R}^n$ to be feasible in (P), g_i is active precisely when $g_i(x) = 0$. We denote the set of all active constraints as:

$$\mathcal{A}_x = \{i : g_i(x) = 0\}.$$ ⌟

Consider the following problem with some simple linear constraints.

Example VI.1.4. Consider the following constrains:

$$\begin{aligned} x_1 + x_2 - 1 &\leq 0, \\ -x_1 &\leq 0, \\ -x_2 &\leq 0. \end{aligned}$$

We can simply draw the region of the feasible region:

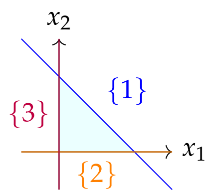


Figure VI.1. Feasible region with respective $\mathcal{A}_x$.

Note that on this graph, the lines (but not intersections) have $|\mathcal{A}_x| = 1$, and the intersections have $|\mathcal{A}_x| = 2$. Likewise, we can also consider the tetrahedron case:

$$\begin{aligned} x_1 + x_2 + x_3 - 1 &\leq 0, \\ -x_1 &\leq 0, \\ -x_2 &\leq 0, \\ -x_3 &\leq 0. \end{aligned}$$

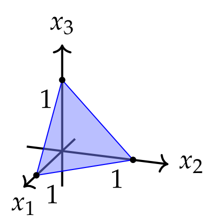


Figure VI.2. Feasible region with respective $\mathcal{A}_x$.

Note that on this graph, the planes (but not line intersections) have $|\mathcal{A}_x| = 1$, the intersection line (but not point intersections) have $|\mathcal{A}_x| = 2$, and the point intersections have $|\mathcal{A}_x| = 3$. ◊

Remark VI.1.5. When evaluating the constraints (with continuous differentiability), a constraint itself will be a $n - 1$ dimensional manifold, and constraint intersecting will be lower dimensional manifolds. ⌟

Lemma VI.1.6. If feasible x in (P) is a local minimum of (P), then there does not exists a direction $d \in \mathbb{R}^n$ which is descent direction for f and g_i for all $i \in \mathcal{A}_x$ simultaneously.

Proof. For the sake of contradiction, suppose that there exists a direction $d \in \mathbb{R}^n$ which is descent direction for f and g_i for all $i \in \mathcal{A}_x$ simultaneously.
There exists some $\delta > 0$ to be small enough such that:

- for all $i \in \mathcal{A}_x$, $g_i(x + \delta d) < g_i(x) = 0$, and
- for all $i \notin \mathcal{A}_x$, $g_i(x + \delta d) < 0$

so that $x + \delta d$ is still feasible.
In fact, we can just construct this δ through the following approaches.

(i) For each $i \notin \mathcal{A}_x$, we can pick δ_i such that for all $y \in N_{\delta_i}(x)$, $|x - y| < \frac{g_i(x)}{2}$, hence we have $g_i(y) < 0$ still.

(ii) Then, since the number of i's is finite, we can just let:

$$\delta := \min_{i \in \mathcal{A}_x} \delta_i.$$

Hence, by this construction, $f(x + \delta d) < f(x)$, so x is not a local minimum. □

Then, we can give a "matrix-fication" version of the previous lemma.

Remark VI.1.7. With the same setup, there does not exists any $d \in \mathbb{R}^n$ such that $J^\intercal d < 0$ for matrix:

$$J^\intercal = \begin{pmatrix} \nabla f^\intercal(x) \\ \nabla g^\intercal_{\mathcal{A}_x^1} x \\ \vdots \\ \nabla g^\intercal_{\mathcal{A}_x^{|\mathcal{A}_x|}} x \end{pmatrix}$$

Note that this rows are just the active components of the gradients. ⌟

Then, we would apply the Gordon's theorem on the matrix $J^\intercal$.

Proposition VI.1.8. If feasible x in (P) is a local minimum of (P), then there exists nonzero $\lambda \geq 0$ such that $J\lambda = 0$.

Following this, we can make this into a more formal form.

Corollary VI.1.9. If feasible x in (P) is a local minimum of (P), then there exists $\beta \in \mathbb{R}$ and $\lambda \in \mathbb{R}^m$ such that $\binom{\beta}{\lambda} \geq 0$ is nonzero such that:

$$\begin{pmatrix} \nabla f(x) & \nabla g_1(x) & \nabla g_2(x) & \cdots & \nabla g_m(x) \end{pmatrix} \begin{pmatrix} \beta \\ \lambda \end{pmatrix} = 0,$$

and such that for each i where $g_i(x) \neq 0$, it holds that $\lambda_i = 0$.

Remark VI.1.10. For the above corollary, it is equivalent to state that for all i, we have $\lambda_i g_i(x) = 0$, which is equivalently by stating that:

$$\sum_{i=1}^{m} \lambda_i g_i(x) = 0.$$

⌟

Then, we will be able to obtain the following result.

Theorem VI.1.11. Fritz-John Optimality Conditions.
If x feasible in (P) is a local minimum of (P), then there exist $\lambda \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$ such that:

$$\begin{aligned} &\beta \nabla f(x) + \nabla g(x) \lambda = 0, \\ &\lambda^\intercal g(x) = 0, \\ &\lambda \geq 0, \beta \geq 0, \quad \text{and} \quad \begin{pmatrix} \beta \\ \lambda \end{pmatrix} \neq 0. \end{aligned}$$

Consider the equations, we can effectively format it as:

$$\begin{aligned} &\beta \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0, \\ &\lambda_i g_i(x) = 0 \qquad \text{for all } i, \\ &\lambda_i \geq 0 \text{ for all } i, \beta \geq 0, \quad \text{and} \quad \begin{pmatrix} \beta \\ \lambda \end{pmatrix} \neq 0. \end{aligned}$$

Remark VI.1.12. Here, we call $\nabla g(x)$ is the Jacobian matrix, *i.e.*:

$$\nabla g = \begin{pmatrix} \nabla g_1(x) & \nabla g_2(x) & \cdots & \nabla g_m(x) \end{pmatrix}.$$

⌟

VI.2 Karush-Kuhn-Tucker Condition

Now, we are equipped with sufficient backgrounds to the final Karush-Kuhn-Tucker (KKT) Condition.

Theorem VI.2.1. Karush-Kuhn-Tucker Condition.
Suppose x feasible in (P) is a local minimum of (P) and if x satisfies the constraint qualification (CQ) that $\{\nabla g_i(x)\}_{i\in\mathcal{A}x}$ are linearly independent.
Then, there exists $\lambda \in \mathbb{R}^m$ such that:

$$\begin{aligned}&\nabla f(x) + \nabla g(x)\lambda = 0,\\ &\lambda^\intercal g(x) = 0,\\ &\lambda \geq 0.\end{aligned}$$

Remark VI.2.2. Here, we consider the existence of $\lambda_1, \cdots, \lambda_m \in \mathbb{R}^m$ being the KKT-multipliers, such that:

$$\begin{aligned}&\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0,\\ &\lambda_i g_i(x) = 0 \qquad \text{for all } i,\\ &\lambda_i \geq 0 \qquad \text{for all } i.\end{aligned}$$

⌟

Proof. Directly follows from Fritz-John Optimality condition (Theorem VI.1.11). By the linear independence of $\{\nabla g_i(x)\}_{i\in\mathcal{A}x}$, it forbids β being zero as well as the nontrivial combination of active $\nabla g_i(x)$ being zero.
This without loss of generality, we can scale $\binom{\beta}{\lambda}$ by $\frac{1}{\beta} > 0$ and they will still satisfy the Fritz-John Optimality condition.
Now, we have $\beta \leftarrow 1$, which we still manage to satisfy $\binom{\beta}{\lambda}$ being nonzero as $\beta = 1$. □

Recall our problem, we have:

$$\begin{aligned}&\min f(x),\\ &\text{s.t. } g(x) \leq 0.\end{aligned} \tag{P}$$

Note that the KKT condition is a necessary condition, but not sufficient condition, for optimality.

Then, we will consider various examples with applying the KKT condition.

Example VI.2.3. Consider the problem:

$$\begin{aligned}&\min e^{-\frac{x_1+x_2}{7}},\\ &\text{s.t. } x_1^2 + x_2^2 - 25 \leq 0,\\ &2x_2 - 3 \leq 0,\\ &-x_1 \leq 0.\end{aligned} \tag{P}$$

First of all, let's visualize the feasible region in $\mathbb{R}^2$:

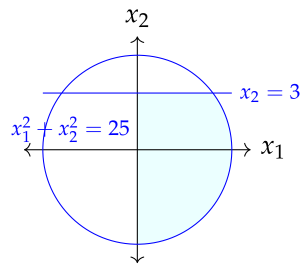


Figure VI.3. Feasible Region in given problem.

Then, we can compute the gradient as:

$$\nabla f(x_1, x_2) = \begin{pmatrix} -\frac{1}{7}e^{-\frac{1}{7}(x_1+x_2)} \\ -\frac{1}{7}e^{-\frac{1}{7}(x_1+x_2)} \end{pmatrix}.$$

Then, we consider the gradient of each active constraint:

$$\nabla g_1 = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}, \qquad \nabla g_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \nabla g_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Here, we can consider the point $(4, 3)$, which has g_1 and g_2 active, which turns out to satisfy that:

$$\nabla g_1(4,3) = (8,6) \qquad \text{and} \qquad \nabla g_2(4,3) = (0,1),$$

which are linearly independent. We can then seek for $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that:

$$\begin{pmatrix} -\frac{1}{7e} \\ -\frac{1}{7e} \end{pmatrix} + \lambda_1 \begin{pmatrix} 8 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0,$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}^\intercal \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} = 0.$$

By observation, we have $\lambda_3 = 0$, and we can solve for $\begin{pmatrix} 8 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7e} \\ \frac{1}{7e} \end{pmatrix}$, which solves into $\lambda_1 = \frac{1}{56e}$ and $\lambda_2 = \frac{1}{28e}$.

Hence, we know that $(4, 3)$ satisfies the KKT condition with KKT-multipliers as $\left(\frac{1}{56e}, \frac{1}{28e}, 0\right)$. ◊

Then, we will provide a case in which a KKT point could appear in the interior.

Remark VI.2.4. KKT in Interior.

If the KKT point is in the interior of the feasible region, we know that it has no active constraint (which is fine for being linearly independent, vacuously). This implies that all the KKT-multipliers must be zero, pushing that $\nabla f(x) = 0$, which implies that the point itself is stationary. ⌟

Then, we will continue to inspect if there are less active conditions.

Example VI.2.5. Continued from Example VI.2.3....

Now, we consider the marked red arc below:

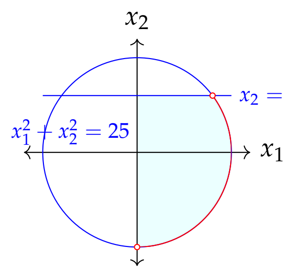


Figure VI.4. Consider the arc on the region looking for KKT condition.

Here, we notice that only the first condition is active, and we notice that $\nabla g_1 = 0$ if and only if $x_1 = x_2 = 0$, which is not possible on the arc, so we have $\lambda_1 \geq 0$ such that:

$$\begin{pmatrix} \frac{1}{7}e^{-\frac{1}{7}(x_1+x_2)} \\ \frac{1}{7}e^{-\frac{1}{7}(x_1+x_2)} \end{pmatrix} = \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix},$$

this implies that $x_1 = x_2$, but this is not possible in any of the cases. ♢

Remark VI.2.6. When there is only one active constraint and it is nonzero, this forces the vector in the opposite direction of the gradient, *i.e.*, the steepest descent direction. ⌟

Example VI.2.7. Another Continued from Example VI.2.3....
Now, we consider the marked red line below:

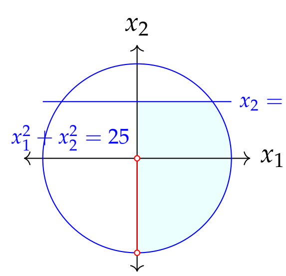


Figure VI.5. Consider the arc on the region looking for KKT condition.

Here, we notice that only the first condition is active, and so $\nabla g_1 = 0$ if and only if $x_1 = x_2 = 0$, which is not possible on the arc, so we have $\lambda_1 \geq 0$ such that:

$$\begin{pmatrix} \frac{1}{7}e^{-\frac{1}{7}(x_1+x_2)} \\ \frac{1}{7}e^{-\frac{1}{7}(x_1+x_2)} \end{pmatrix} = \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix},$$

this implies that $x_1 = x_2$, but this is not possible in any of the cases. ♢

In the above example, we can notice that we have the problem with an additional condition on the feasible region $S \subset \mathbb{R}^n$ as an open set. As we consider x as a local minimum, plus the constraint qualifications, then there exists $\lambda_1, \cdots, \lambda_m$ in which:

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0, \qquad \lambda_i g_i(x) = 0 \text{ for all } i, \qquad \text{and} \qquad \lambda_i \geq 0 \text{ for all } i.$$

Then, we consider another problem.

Example VI.2.8. Let the optimization problem be:

$$\begin{aligned} &\min\ x^2 + y^2, \\ &\text{s.t. } x \geq 1. \end{aligned}$$

We can visualize the feasible region as follows:

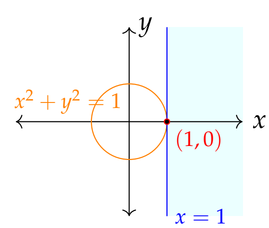


Figure VI.6. Feasible region and optimal level curve in given problem.

Now, we wish to find all KKT points. We rewrite the problem as:

$$\begin{aligned} &\min\ x^2 + y^2, && (f \leftarrow x^2 + y^2) \\ &\text{s.t. } 1 - x \leq 0. && (g \leftarrow 1 - x) \end{aligned}$$

Thus, we have that:

$$\nabla f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \text{and} \qquad \nabla g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Now, let's consider the two cases with the KKT condition:

(i) When g is not active, *i.e.*, $x > 1$, we have the constraint qualification vacuously satisfied, but if we want $\nabla f \binom{x}{y} = 0$ is not possible on the feasible region, so there is KKT point here.

(ii) When g is active, the constraint qualification is composed of $\{\binom{-1}{0}\}$ linearly independently, and consider the KKT condition in which:

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0,$$

with $\lambda \geq 0$, this pushes $y = 0$, hence the only possible KKT point is $(1, 0)$.

With the KKT condition satisfied only at $(1, 0)$. ♢

Now, let's consider the problem back into linear programming:

Remark VI.2.9. KKT in Linear Programs.

Consider the problem with $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ that:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t.}\ & Ax \geq b, \\ & x \geq 0. \end{aligned} \tag{LP}$$

Just not to forget, its dual problem is for $y \in \mathbb{R}^m$ that:

$$\begin{aligned} \max\ & b^\intercal y, \\ \text{s.t.}\ & A^\intercal y \geq c, \\ & y \geq 0. \end{aligned} \tag{DP}$$

Then, we think about the problem as a generalized program:

$$\begin{aligned} \min\ & c^\intercal x, \\ \text{s.t.}\ & \begin{pmatrix} -A \\ -\operatorname{Id} \end{pmatrix} x + \begin{pmatrix} b \\ 0 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{P}$$

Hence, we would take the gradients, respectively, as:

$$\nabla f = c \qquad \text{and} \qquad \nabla g^\intercal = \begin{pmatrix} -A^\intercal & -\operatorname{Id} \end{pmatrix}.$$

Then, we will connect the KKT conditions here:

(i) Consider the first one as $\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) = 0$, in which we have:

$$c + \begin{pmatrix} -A^\intercal & -\operatorname{Id} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0,$$

in which we divide λ into the parts associated with $-A^\intercal$ and with $-\operatorname{Id}$. Thence, we can derive that:

$$A^\intercal y + z = c \iff = c - A^\intercal y.$$

(ii) Then, we consider the condition that $\lambda_i g_i(x) = 0$, so we have that:

$$\begin{aligned} y^\intercal(b - Ax) &= 0, & y &\geq 0, \\ z^\intercal(-x) &= 0, & z &\geq 0. \end{aligned}$$

Here, we can equivalently write that:

$$y^\intercal(Ax - b) = 0 \qquad \text{and} \qquad z^\intercal x = 0.$$

(iii) Eventually, since $\lambda_i \geq 0$, we have $y \geq 0$ and $z \geq 0$.

Therefore, by combining (i) and (iii), we have:

$$A^\intercal y \leq c \qquad y \geq 0,$$

and also:

$$c - A^\intercal y \geq 0,$$

which rules out all about z, so we have:

- (Assumption). Primal feasibility of x:

$$Ax \geq b \qquad x \geq 0.$$

- (i and iii). Dual feasibility of y:

$$A^\intercal y \leq c, y \geq 0.$$

- (ii and iii). Complimentary slackness:

$$x^\intercal(x - A^\intercal y) = 0 \text{ and } y^\intercal(Ax - b) = 0,$$

  where $x \geq 0$, $y \geq 0$, $x - A^\intercal y \geq 0$, and $y^\intercal(Ax - b) = 0$. ⌟

Notice that the linear programing is convex, and for more general convex problems.

Remark VI.2.10. If the problem is convex, the KKT condition would be sufficient. ⌟

Previously, we have been working on the necessary conditions as KKT, *i.e.*, if it does not satisfy, the point is not a KKT point, and now we will develop a sufficient condition on an additional condition – convexity.

Theorem VI.2.11. Suppose $S \subset \mathbb{R}^n$ is nonempty, open, and convex set and $f, g_1, \cdots, g_m : S \to \mathbb{R}$ being continuously differentiable, and convex functions. We give the problem:

$$\begin{aligned} \min\ & f(x), \\ \text{s.t. } & g_1(x) \leq 0, \\ & g_2(x) \leq 0, \\ & \vdots \\ & g_m(x) \leq 0, \\ & x \in S. \end{aligned} \tag{P}$$

If $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ satisfies the KKT conditions (note that there is no need for the constraint qualification), *i.e.*:

- $x \in S$, for all i, $g_i(x) \leq 0$,
- $\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) = 0$, for all i, $\lambda_i g_i(x) = 0$ and $\lambda_i \geq 0$,

then x is a global minimum of (P).

Now, we think about am example of optimization conditions.

Remark VI.2.12. Active Constraints.

Consider the optimization problem:

$$\begin{aligned} &\min f(x), \\ &\text{s.t. } g_1(x) \leq 0, \cdots, g_m(x) \leq 0, \\ &\quad h_1(x) = 0, \cdots, h_\ell(x) = 0, \end{aligned} \tag{P}$$

where $f, g_1, \cdots, g_m, h_1, \cdots, h_\ell : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable.
Also, we shall notice that the equality conditions for $h_i(x) = 0$ can be modified into:

$$h_i(x) = 0 \quad \Longleftrightarrow \quad h_i(x) \leq 0 \text{ and } h_i(x) \geq 0.$$

Therefore, we can consider the KKT conditions:

- $\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) + \sum_{i=1}^\ell \mu_i' \nabla h_i(x) + \sum_{i=1}^\ell \mu''\big(-\nabla h_i(x)\big) = 0,$
- for all i, $\lambda_i g_i(x) = 0$, $\mu_i' h_i(x) = 0$, and $-\mu_i'' h_i(x) = 0$, and
- for all i, $\lambda_i \geq 0$, $\mu_i' \geq 0$, $\mu_i'' \geq 0$.

Notice that for the following pairs:

$$h_i(x) = 0 \quad \Longleftrightarrow \quad h_i(x) \leq 0 \text{ and } h_i(x) \geq 0.$$

we would know that $h_i(x) = 0$ which means that the constraints about h_i's are always active, so we can simplify the KKT condition into:

- $\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) + \sum_{i=1}^\ell \mu_i \nabla h_i(x) = 0,$
- for all i, $\lambda_i g_i(x) = 0$, and
- for all i, $\lambda_i \geq 0$.

⌟

Then, lets look at an example for linear program.

Example VI.2.13. Consider the following problem:

$$\begin{aligned} &\min \frac{1}{2} x^\intercal Q x + c^\intercal x + d, \\ &\text{s.t. } Ax = b, \end{aligned}$$

where we have $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. The problem can be equivalently written as:

$$\begin{aligned} &\min \frac{1}{2} x^\intercal Q x + c^\intercal x, \\ &\text{s.t. } Ax - b = 0. \end{aligned}$$

Note that the problem is linear, since we have the Hessian Q being positive definite everywhere, so we have a convex program.
For the KKT condition, we have:

- Feasibility: $Ax = b$,
- Constraint: $Qx + c + A^\intercal \mu = 0$.

Here, we can express this in terms of:

$$\begin{pmatrix} Q & A^\intercal \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} -c \\ b, \end{pmatrix}$$

and here if A is full rank (without loss of generality), then the matrix is invertible and we have the solution:

$$\begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} Q & A^\intercal \\ A & 0 \end{pmatrix} \begin{pmatrix} -c \\ b \end{pmatrix}.$$

◊

Then, we will consider some examples with applying the KKT conditions:

Example VI.2.14. Consider the optimization problem:

$$\begin{aligned} &\min x, \\ &\text{s.t. } 1 - x^2 - y^2 \leq 0. \end{aligned}$$

We want first illustrate this region and some level sets graphically.

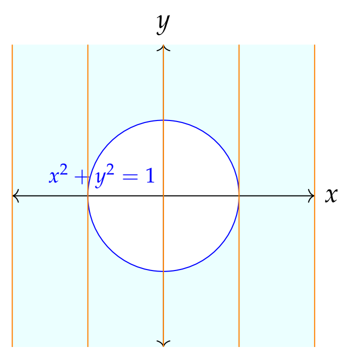


Figure VI.7. Feasible regions and level sets.

Here, we first find the gradients of the program and constraints:

$$\nabla f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \nabla g = \begin{pmatrix} -2x \\ -2y \end{pmatrix}.$$

Here, we can observe that $\binom{1}{0}$ is a KKT point with KKT multiplier $\lambda = \frac{1}{2}$, and consider the descent directions:

- The descent direction for f is $\binom{-1}{y}$ for all y. The steepest descent is $\binom{-1}{0}$.
- The descent direction for g is $\{\binom{x}{y} \neq 0 : x \geq 0\}$. The steepest descent direction is $\binom{2}{0}$.

Here, if we put all directions on the same graph:

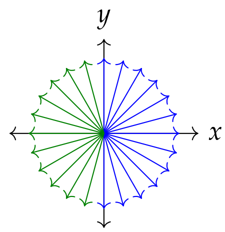


Figure VI.8. Descent direction for f (in green) and for g (in blue).

Therefore, we no that there is no common descent direction, meaning that there is not a linear descent direction for both. However, there could be curved directions. ◊

Then, we are resolving a more elaborated example.

Example VI.2.15. Consider the program:

$$\min \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$
$$\text{s.t. } x_1 + x_2 + 1 \leq 0.$$

Now, we can illustrate the region:

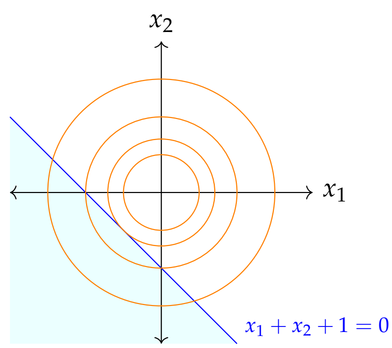


Figure VI.9. Feasible regions and level sets.

Here, we consider the gradients as:

$$\nabla f = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \nabla^2 f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\nabla g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \nabla^2 g = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

To satisfy the KKT condition, we have:

$$\begin{cases} \nabla f(x_1, x_2) + \lambda \nabla g(x_1, x_2) = 0, \\ \lambda \geq 0, \\ x_1 + x_2 + 1 = 0. \end{cases}$$

This implies that $\binom{x_1}{x_2} = -\lambda\binom{1}{1}$ so we have $x_1 + x_2 = -1$, hence we have $x_1 = x_2 = -\frac{1}{2}$, so $\binom{-\frac{1}{2}}{-\frac{1}{2}}$ is the KKT

point with the KKT multiplier $\lambda = \frac{1}{2}$.
This is exactly the intersection between the circle cutting the line for active constraint. ◊

Then, we will be considering a more complicated example:

Example VI.2.16. Consider the optimization problem:

$$\min \frac{1}{2} x^\intercal A x, \quad \text{s.t. } b^\intercal x + c \leq 0,$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, $b \in \mathbb{R}^n$ is nonzero, and $c \in \mathbb{R}$ is positive.
Here, we have the functions and gradients as:

$$f(x) = \frac{1}{2} x^\intercal A x, \qquad \nabla f = Ax, \nabla^2 f = A,$$
$$g(x) = b^\intercal x + c, \qquad \nabla g = v, \nabla^2 g = 0.$$

Hence, we have the KKT condition as:

$$\begin{cases} \nabla f(x) + \lambda \nabla g(x) = 0, \\ \lambda \geq 0, \\ g(x) = 0. \end{cases}$$

Therefore, we have $Ax + \lambda b = 0$, $\lambda \geq 0$, and $b^\intercal x + c = 0$, so we have:

$$x = -\lambda A^{-1} b \qquad \text{and} \qquad b^\intercal(-\lambda A^{-1} b) + c = 0.$$

Here, we have the associated KKT multiplier as:

$$\lambda = \frac{c}{b^\intercal A^{-1} b} > 0$$

since both the numerator and the denominator are positive.
Specifically, for the denominator, this is because A^{-1} is also positive definite from A being positive definite.
Therefore, the KKT point is:

$$x = -\frac{c}{b^\intercal A^{-1} b} A^{-1} b$$

with associated KKT multiplier:

$$\lambda = \frac{c}{b^\intercal A^{-1} b}.$$

◊

From this example, we have seen that doing algebraic manipulations is important.

Then, we will discuss a problem in which we have local positive definiteness leading to local minimization.

Proposition VI.2.17. Local Positive Deiniteness.
Consider the optimization problem:

$$\begin{aligned} \min\ & f(x), \\ \text{s.t. } & g_1(x) \leq 0, \\ & g_2(x) \leq 0, \\ & \vdots \\ & g_m(x) \leq 0. \end{aligned} \tag{P}$$

If $\overline{x}$ is feasible and if $\nabla^2 f(\overline{x})$ is positive definite, and for all $i \in \mathcal{A}(\overline{x})$, $\nabla^2 g_i(x)$ is positive definite. Then, if $\overline{x}$ is a KKT point of (P), $\overline{x}$ is a local minimum of (P).

Proof. By the continuity of f, g_i's, and the Hessians, etc., there exists $\epsilon > 0$ such that:

- for all $i \notin \mathcal{A}(\overline{x})$ and for all $x \in N_\epsilon(\overline{x})$, we have $g_i(x) \leq 0$,
- for all $i \in \mathcal{A}(\overline{x})$ and for all $x \in N_\epsilon(\overline{x})$, we have $\nabla^2 g_i$ being positive (semi-)definite, and
- for all $x \in N_\epsilon(x)$, $\nabla^2 f$ positive (semi-)definite.

Consider the modified problem:

$$\begin{aligned} \min\ & f(x), \\ \text{s.t. } & g_i(x) \leq 0 \text{ for all } i \in \mathcal{A}(\overline{x}), \\ & x \in N_\epsilon(\overline{x}). \end{aligned} \tag{$\hat{\text{P}}$}$$

Here, on $N_\epsilon(\overline{x})$, we have $f(x)$ being convex, and on $N_\epsilon(\overline{x})$ we have $g_i(x)$ being convex, and $N_\epsilon(\overline{x})$ is a open, convex set. Hence, we have a Convex problem.
If $\overline{x}$ is KKT point of (P), then it is a KKT point of ($\hat{\text{P}}$). Note that for the inactive constraints, we automatically have the KKT multipliers being zero.
Therefore, $\overline{x}$ is the KKT point to a convex program ($\hat{\text{P}}$), which implies that $\overline{x}$ is a global minimum for ($\hat{\text{P}}$), and ($\hat{\text{P}}$) is a restriction of (P) to a neighborhood of $\overline{x}$, so $\overline{x}$ is a local minimum of (P). □

Remark VI.2.18. For all the (semi-)positive definite and convex in the proof, we can replace them via positive definite and strictly convex, then we can show uniqueness for $\overline{x}$ being local minimum. ⌟

As a side note, when the dimension is low (*i.e.*, $n = 2$), we can try to analyze the KKT point through graphical analysis, namely with the following example:

Example VI.2.19. Graphical Anlaysis for KKT condition.
Consider the problem:

$$\begin{aligned} \min\ & e^{x_2}, \\ \text{s.t. } & x_1^2 \leq x_2. \end{aligned}$$

We can visualize the graph as follows:

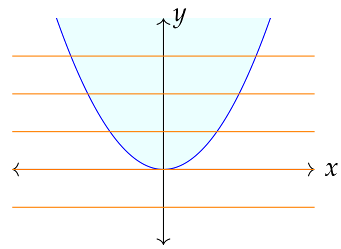


Figure VI.10. Feasible regions and level sets.

Then, we can take the derivatives as:

$$f(x) = e^{x_2}, \qquad \nabla f(x) = \begin{pmatrix} 0 \\ e^{x_2} \end{pmatrix}, \qquad \nabla^2 f(x) = \begin{pmatrix} 0 & 0 \\ 0 & e^{x_2} \end{pmatrix},$$

$$g(x) = x_1^2 - x_2, \qquad \nabla g(x) = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, we have a convex problem, and to observe that the KKT point is:

$$\begin{pmatrix} 0 \\ e^0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\binom{0}{0}$ is a KKT point with KKT multiplier $\lambda = 1$, hence it is the global minimum. $\Diamond$

VI.3 Newton's Method, Revisit

We are going to revisit the Newton's method, for root finding. In the previous chapter (V.3), we are also really using the method to *find the root of derivatives*.

Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is component-wise continuously differentiable. We want to seek a "zero" of F, *i.e.*, $x^* \in \mathbb{R}^n$ such that $F(x^*) = 0$.

Algorithm VI.3.1. Finding Roots.

Start with $x^{(0)} \leftarrow \mathbb{R}^n$ and $k \leftarrow 0$.
while $F\big(x^{((k))}\big)$ is not sufficiently close to 0 **do**
  Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be linear approximation of F about $x^{(k)}$.
  Let $x^{(k+1)}$ to be defined such that $G\big(x^{(k+1)}\big) = 0$.
  Let $k \leftarrow k + 1$.
end while

Here, we note that for all i, we have:

$$F_i(x) \approx F_i\big(x^{(k)}\big) + \nabla F_i^{\mathsf{T}}\big(x - x^{(k)}\big) =: G_i(x).$$

Hence, if we combine the entries together:

$$F(x) \approx F\big(x^{(k)}\big) + \underbrace{\begin{pmatrix} \nabla F_1^\intercal\big(x^{(k)}\big) \\ \vdots \\ \nabla F_n^\intercal\big(x^{(k)}\big) \end{pmatrix}}_{\nabla^\intercal F\left(x^{(x)}\right)} \big(x - x^{(k)}\big) =: G(x),$$

in denoting $\nabla^\intercal F\big(x^{(k)}\big)$ to be the **Jacobian**, hoping that it could be invertible.
Therefore, $x^{(k+1)}$ is the vector such that the above equation equates to 0, and with the assumption that the Jacobian is invertible:

$$x^{(k+1)} = x^{(k)} \underbrace{-\big(\nabla F^\intercal\big(x^{(k)}\big)\big)^{-1} F\big(x^{(k)}\big)}_{\text{Newton's step/direction}}.$$

Remark VI.3.2. Solving Linear Systems.
We, at many times just solve for the vector $\big(x^{(k+1)} - x^{(k)}\big)$ with the problem:

$$\nabla F^\intercal\big(x^{(k)}\big)\big(x^{(k+1)} - x^{(k)}\big) = -F\big(x^{(k)}\big),$$

since the high dimensional matrices could be hard to solve. ⌟

Example VI.3.3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. For optimizing f (minimum or maximum), we seek stationary point of gradient, *i.e.*, we seek $x^* \in \mathbb{R}^n$ such that $\nabla f(x^*) = 0$.
Indeed, $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a high dimensional function, and we are seeking its root. Now, we can consider the Jacobian of ∇f (transposed) as $\nabla^2 f$, and hence solving it becomes:

$$x^{(k+1)} = x^{(k)} - \big(\nabla^2 f(x^{(k)})\big)^{-1} \nabla f(x^{(k)}),$$

which aligns with the characterization in Section V.3, *i.e.*, the Newton's method for gradient descent is a form of Newton's method for root finding. ♢

VI.4 Quadratic Programming

We introduce interior point method to solve quadratic programs (which also solves linear programs).

Definition VI.4.1. Quadratic Program.
We define a general form of the Quadratic Program that:

$$\begin{aligned} \min\ & \frac{1}{2} x^\intercal Q x + c^\intercal x, \\ \text{s.t. } & Ax = b, \\ & x \geq 0, \end{aligned} \tag{QP}$$

where $a \in \mathbb{R}^{m \times b}$ is full row rank, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. ⌟

Remark VI.4.2. Technically, if we lift the condition of Q to be symmetric and positive semidefinite, allowing $Q = 0$ will make it a linear program. ⌟

Given the linear constraints, the constraints are convex, and since the Hessian Q is positive definite, so it is also convex. Therefore, KKT condition is sufficient for global optimality.

We now equivalently format the problem as:

$$\begin{aligned} \min\ & \frac{1}{2}x^\intercal Q x + c^\intercal x, \\ \text{s.t. } & b - Ax = 0, \\ & -\operatorname{Id} x \leq 0, \end{aligned} \tag{QP}$$

The KKT condition is given by:

$$Ax = b \qquad \text{and} \qquad x \geq 0,$$

additionally with:

$$Qx + c + \begin{pmatrix} -A^\intercal & -\operatorname{Id} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0,$$

where we have $y \in \mathbb{R}^n$, $z \in \mathbb{R}^n$, which are the KKT values (or *KKT multipliers*), and

$$z^\intercal x = 0 \qquad \text{and } z \geq 0.$$

for complimentary slackness (since y is automatically active).

Remark VI.4.3. Primal Feasibility and KKT.
With the above setup, we can consider $F : \mathbb{R}^{n+m+n} \to \mathbb{R}^{n+m+n}$, so we have:

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -Qx - c + A^\intercal y + z \\ Ax - b \\ z \odot x, \end{pmatrix}$$

where $z \odot x$ (or sometimes denoted $z \circ x$) is the component-wise multiplication (**Hadamard product**, if calling it *fancily*), and $x, z \geq 0$.
We would wish to find an algorithm to find the solutions to $F(x^*, y^*, z^*) = 0$, which is root finding for F, while maintaining $x^* \geq 0$ and $z^* \geq 0$.
Note that by this convex program, the existence of such x^* implies **global minimum** and $\binom{y^*}{z^*}$ forms the KKT multipliers. ⌟

To utilize the root finding techniques, we would use the gradient here:

$$\nabla F^\intercal \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -Q & A^\intercal & \operatorname{Id} \\ A & 0 & 0 \\ \operatorname{diag}(z) & 0 & \operatorname{diag}(x) \end{pmatrix},$$

where for any $v \in \mathbb{R}^n$ we denote $\operatorname{diag}(v) = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_n \end{pmatrix} \in \mathbb{R}^{n \times n}$.

Remark VI.4.4. The above matrix $\nabla F^\intercal(x, y, z)$ is invertible if $x > 0$ and $z > 0$. ⌟

Thus, to conduct a Newton's step, we have:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} - \underbrace{\left[\nabla F^\intercal \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} \right]^{-1} F \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}}_{\begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix}, \textit{ i.e.}, \text{ Newton's direction}}.$$

In practice, we will solve the following linear system:

$$\nabla F^\intercal \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} \begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix} = -F \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix},$$

and then we have:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} + \begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix}.$$

However, this naturally induce a problem:

even if $x^{(k)} \geq 0$ and $z^{(k)} \geq 0$, it could be possible that $x^{(k+1)} \ngeq 0$ or $z^{(k+1)} \ngeq 0$.

Remark VI.4.5. Naïve Newton.
We assume that $x^{(k)} > 0$ and $z^{(k)} > 0$, we can then choose $\alpha^{(k)} \in (0, 1]$:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} + \alpha^{(k)} \begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix}.$$

However, the problem is that a step sizes could go towards zero. ⌟

VI.5 The Interior Point Method

Then, we would want to find a "good" approximation as a guide for the function.

Definition VI.5.1. For all $\tau > 0$, we consider $F_\tau : \mathbb{R}^{n+m+n} \to \mathbb{R}^{n+m+n}$, we have:

$$F_\tau \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -Qx - c + A^\intercal y + z \\ Ax - b \\ z \odot x - \tau \vec{1}, \end{pmatrix}$$

This new definition turns out to be useful, as we have the following theorem.

Theorem VI.5.2. If there exists $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $F_{1:n+m} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0_{n+m}$ and $x > 0$ while $z > 0$, then for all $\tau > 0$, there exists a unique $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $F_\tau \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ while $x \geq 0$ and $y \geq 0$.

We call this root $\begin{pmatrix} x_\tau \\ y_\tau \\ z_\tau \end{pmatrix}$ and $x_\tau \to x^*$ as the solution of the quadratic programming as $\tau \to 0$.

Here, we consider the set:

$$\left\{ \begin{pmatrix} x_\tau \\ y_\tau \\ z_\tau \end{pmatrix} : \tau > 0 \right\},$$

which is called the **central path**.

Definition VI.5.3. For $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^{n+m+n}$ such that $x > 0$ and $z > 0$. The average complementarity is defined as:

$$\beta := \frac{z^\intercal x}{n}.$$

For fixed parameter $\gamma \in (0, 1)$:

$$N_2(\gamma) := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^{n+m+n} : x > 0, z > 0, \text{ and } \left\| z \circ x - \frac{z^\intercal x}{n} \vec{1} \right\|_2 \leq \gamma \frac{z^\intercal x}{n} \right\}.$$

Remark VI.5.4. Envelopes.
The above set $N_2(\gamma)$ can be considered as a *safety envelope* that gradually squeeze to the central path on the boundary.

For fixed parameter $\delta \in (0, 1)$:

$$N_{-\infty}(\delta) := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^{n+m+n} : x > 0, z > 0, \text{ and for all } i, z_i x_i \geq \delta \frac{z^\intercal x}{n} \right\}.$$

Then, we can consider a path following algorithms.

Algorithm VI.5.5. Path Following Algorithm.

Choose $\epsilon_{\min}, \epsilon_{\max}$ such that $0 < \epsilon_{\min} < \epsilon_{\max} < 1$.
If we use "short path" version, choose $\gamma \in (0, 1)$ and $(x^{(0)}, y^{(0)}, z^{(0)}) \in N_2(\gamma)$.
If we use "long step" version, choose $\delta \in (0, 1)$ and $(x^{(0)}, y^{(0)}, z^{(0)}) \in N_2(\gamma)$.
for $k \leftarrow 0, 1, \cdots$ **do**
  Choose $\epsilon^{(k)} \in [\epsilon_{\min}, \epsilon_{\max}]$, set $\beta^{(k)} \leftarrow \frac{(z^{(k)})^\intercal x^{(k)}}{n}$.
  Solve for the linear system that:

$$\left[\nabla F^\intercal \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} \right] \begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix} = -F_{\epsilon^{(k)}\beta^{(k)}} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}$$

  Choose $\alpha^{(k)} > 0$ as close to 1 as possible such that:

$$\begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} + \alpha^{(k)} \begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix} \in \begin{cases} N_\delta(\gamma), & \text{if in short path,} \\ N_{-\infty}(\delta), & \text{if in long step.} \end{cases}$$

  Set $\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} \leftarrow \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} + \alpha^{(k)} \begin{pmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \\ \Delta z^{(k)} \end{pmatrix}$.
end for

We can still consider this as a part of linear programming.

Remark VI.5.6. Consider the condition that $z = c - A^\intercal y$, $z \geq 0$, and $z \perp x$, with $Ax = b$ and $x \geq 0$, we then have:

$$A^\intercal y \leq x \qquad \text{and} \qquad c - A^\intercal y \perp x.$$

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