AS.110.417: Partial Differential Equations

Lecture Notes

James Guo

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Contents

Acknowledgments:

- The Lecture Notes records the course contents for `AS.110.417` Partial Differential Equations instructed by *Dr. Xiong Wang* at *Johns Hopkins University*.
- The notes summaries the lecture contents, notes, and adapted contents from the following sources:
  - *Elementary Applied Partial Differential Equation* by Richard Haberman.
  - *Partial Differential Equations in Action* by Salsa Sandro and Gianmaria Verzini.
- The notes is a summary of the lectures, and it might contain minor typos or errors. Please point out any notable error(s).

Best regards, James Guo.
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1 Preliminaries

1.1 PDEs as Mathematical Models

PDEs are mathematical models that follows:

(i) General laws: such as conservation laws or balance laws;

(ii) Constitutive relations are more specific or experimental nature or feature: such as Fourier laws.

The outcome of the laws results in a PDE or a system of PDEs, in the form of:

$$F(x_1, x_2, \cdots, x_n, \cdots, u_{x_1}, u_{x_2}, \cdots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \cdots, u) = 0.$$

Remark 1.1.1. Classifications on PDEs.

Different PDEs are classified based on the following criteria:

(i) The highest order of differentiation is the order of PDEs;

(ii) A PDE is linear if F is *linear* on u and its derivatives, otherwise it is *non-linear*.

1.2 Gauss Divergence Theorem

Theorem 1.2.1. Gauss Divergence Theorem.

Let W be a symmetric elementary region (or *good* region). Denote ∂W as the oriented closed surface that bounds W, then:

$$\iiint_W \nabla \cdot \mathbf{F} dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S},$$

or alternatively:

$$\iiint_W (\operatorname{div} \mathbf{F}) dV = \iint_{\partial W} (\mathbf{F} \cdot \mathbf{n}) dS.$$

The Gaussian Divergence Theorem relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed.
More importantly, the divergence following some properties in vector calculus.

Proposition 1.2.2. Properties of Vector Calculus.

Let φ be a scalar valued function, $\mathbf{F}$ and $\mathbb{G}$ be vector fields with a and b as real numbers, the following properties hold for true for divergence operator:

(i) Linearity: $\nabla \cdot (a\mathbf{F} + b\mathbf{G}) = a\nabla \cdot \mathbf{F} + b\nabla \cdot \mathbf{G}$;

(ii) Product rule: $\nabla \cdot (\varphi \mathbf{F}) = (\nabla \varphi) \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F})$;

(iii) Laplacian of vector field: $\Delta \varphi = \nabla \cdot (\nabla \varphi)$;

(iv) Cross product in $\mathbb{R}^3$: If $\mathbf{F}$ and $\mathbf{G}$ has codomain $\mathbb{R}^3$, then $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$;

(v) Curl of the gradient in $\mathbb{R}^3$: If φ has domain $\mathbb{R}^3$, then $\nabla \cdot (\nabla\varphi) = 0$.

(vi) Divergence and curl in $\mathbb{R}^3$: If $\mathbf{F}$ has codomain $\mathbb{R}^3$, then $\nabla \cdot (\nabla \cdot \mathbf{F}) = 0$.

(vii) Homological relationships on ∇ operator in $\mathbb{R}^3$: The following chain complex is exact:

$$\text{Scalar Field} \xrightarrow{\text{grad}} \text{Vector Field} \xrightarrow{\text{curl}} \text{Vector Field} \xrightarrow{\text{div}} \text{Scalar Field}$$

With the properties of *Guass Divergence Theorem* and *Properties of Vector Calculus*, certain properties in single variable calculus applies for vector calculus.

Corollary 1.2.3. Properties on Integration for Vector Valued Function.

For a *good* region W, let φ, ψ be scalar valued function and $\mathbf{F}$ be vector field in $\mathbb{R}^3$, them the following properties hold:

(i) Integration by parts formula:
$$\iiint_W (\nabla\varphi) \cdot \mathbf{F} dV = \iint_{\partial W} \varphi(\mathbf{F} \cdot \mathbf{n}) dS - \iiint_W \varphi(\nabla \cdot \mathbf{F}) dV;$$

(ii) Green's first identity:
$$\iint_{\partial W} \varphi(\nabla\psi \cdot \mathbf{n}) dS = \iiint_W [\varphi(\Delta\psi) + (\nabla\varphi) \cdot (\nabla\psi)] dV;$$

(iii) Green's second identity:
$$\iint_{\partial W} [\varphi(\nabla\psi) - \psi(\nabla\varphi)] \cdot \mathbf{n} dS = \iiint_W [\varphi(\Delta\psi) - \psi(\Delta\varphi)] dV.$$

1.3 Well-Posed Problems

Definition 1.3.1. Well-posedness.

A problem is *well posed* when it possesses the following properties:

(i) Existence: There exists at least one solution;

(ii) Uniqueness: There exists at most one solution;

(iii) Continuity: The solution depends continuously on the data, *i.e.*, a small error on initial/boundary data entails a small error on the solution.

The well-posedness is useful in mathematical models, as the existence of uniqueness induces intrinsic coherence and stability for a PDE, whereas continuity induces accuracy on the numerical approximations for each PDE.

Establishing suitable conditions in all initial and boundary data with the formulation of the PDE allows the feature of well-posed PDEs. In particular, many PDEs with the suitable initial and boundary conditions are well-posed.

Example 1.3.2. Examples of PDEs.

The following encapsulates several models in PDEs:

(i) Heat equation or diffusion:
$$u_t - \kappa\Delta u = 0;$$

(ii) Wave equation:
$$u_{tt} - c^2\Delta u = 0;$$

(iii) Laplace's or potential equation:
$$\Delta u = 0;$$

(iv) Fisher's equation:
$$u_t - \kappa\Delta u = cu(\mu - u);$$

(v) Burgers equation:
$$u_t + cuu_x = 0.$$

2 Heat Equations

2.1 Heat Equation in 1-D Rod

In deriving the *heat equation* in 1-D Rod, our primary analysis is based on the *conservation of energy*.

Proposition 2.1.1. Conservation of Energy.

Conservation of energy is the conservative law for heat equation, it is described as:

$$\begin{array}{l} \text{rate of change of total} \\ \text{energy in time} \end{array} = \begin{array}{l} \text{heat energy flowing across} \\ \text{boundaries per unit time} \end{array} + \begin{array}{l} \text{heat energy generated} \\ \text{inside per unit time} \end{array}.$$

In deriving the energy conservation, we consider the following aspects:

- Amount of thermal energy per unit volume.
  A 1-D rod should have cross-sectional area A being small and length L being long. Hence, give point a and b on the rod, the there could be segments for $[x, x+\Delta x] \in [a, b]$.

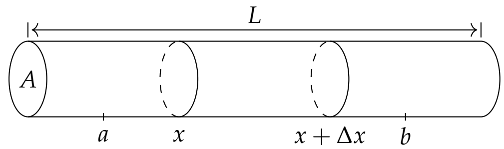

Figure 2.1. Thermal energy through a rod.

Let $e(x,t)$ denote the thermal energy density, the energy for the segment can be approximated as:

$$\text{Energy} \approx \sum_{i=0}^{(b-a)/\Delta x} e(a + i\Delta x, t)\Delta x A,$$

which the total energy from point a to b on the rod tends to:

$$\text{Energy} = \int_a^b e(x,t)A\,dx.$$

- Flow in and flow out.
  Let $\phi(x,t)$ denote the heat flux, *or the amount of energy per unit time flowing to the right per unit surface area*. Then, the heat energy flowing across the boundaries ar a and b per unit time is:

$$\phi(a,t)A - \phi(b,t)A.$$

- Heat sources.
  Let $Q(x,t)$ denote the heat energy generated per unit volume per unit time.

Then, combining the energy conservation, we have:

$$\frac{d}{dt}\int_a^b e(x,t)A\,dx = \phi(a,t)A - \phi(b,t)A + \int_a^b Q(x,t)A\,dx.$$

By applying the fundamental theorem of calculus and the *Gauss Divergence Theorem*, we can have that:

$$\frac{\partial}{\partial t}e(x,t) = -\Delta\phi(x,t) + Q(x,t)$$

Proposition 2.1.2. Fourier's Law.

Let $u(x,t)$ be the temperature, $\rho(x)$ be the mass density, and $c(x)$ be the specific heat, then the basic relationship is:

$$e(x,t) = c(x)\rho(x)u(x,t).$$

Furthermore, the Fourier law states that:

$$\phi = -\kappa_0 \nabla u(x,t).$$

By the *Conservation of Energy* and *Fourier's Law*, we proceed to that:

$$\begin{aligned} \partial_t[c(x)\rho(x)u(x,t)] &= -\nabla\cdot[-\kappa_0\nabla u(x,t)] + Q(x,t) \\ c(x)\rho(x)u_t(x,t) &= \partial_x[\kappa_0\nabla u(x,t)] + Q(x,t) \\ \frac{\partial u(x,t)}{\partial t} &= \kappa\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \end{aligned}$$

where $\kappa = \dfrac{\kappa_0}{c\rho}$ and f is the source.

Theorem 2.1.3. Heat Equation in 1-D Rod.

The standard heat equation is:

$$u_t(x,t) = \kappa\Delta u(x,t) + f(x,t).$$

Some solutions of the standard heat equation are:

$$u(x,t) \equiv 1, \quad u(x,t) = x, \quad u(x,t) = \frac{x}{\sqrt{t}}\exp\left(-\frac{x^2}{t}\right), \quad \cdots$$

To form a well-posed problem, we need the boundary and initial conditions.

2.2 Initial and Boundary Conditions

In 1-D case, we have the problem as:

$$\partial_t u(x,t) = \kappa \partial_{xx} u(x,t) + Q(x,t), \text{ where } x \in (0,L) \text{ and } t > 0.$$

The initial condition would be:

$$u(x,0) = f(x), \text{ with } x \in [0,L].$$

Definition 2.2.1. Boundary Conditions for 1-D Rod.

The 1-D heat equation for a rod $[0,L]$ induces some boundary conditions, encapsulated as follows:

(i) Dirichlet Boundary Condition:

$$u(0,t) = \phi(t), u(L,t) = \psi(t), \text{ with } t > 0;$$

(ii) Neumann Boundary Condition:

$$\partial_x u(0,t) = \phi(t), \partial_x u(L,t) = \psi(t), \text{ with } t > 0;$$

(iii) Robin Boundary Condition:

$$\begin{cases} a_1 \partial_x u(0,t) + a_2 u(0,t) = \phi(t), \\ b_1 \partial_x u(L,t) + b_2 u(L,t) = \psi(t). \end{cases}$$

With each of the initial conditions, it is possible to determine the equilibrium.

Definition 2.2.2. Equilibrium State.

The Equilibrium state is the steady, a balance is achieved due to equal action of opposing forces. For the heat equation for 1-D rod, we define the equilibrium as:

$$u(x) = \lim_{t\to\infty} u(x,t).$$

Definition 2.2.3. Parabolic Boundary.

Let $Q_T = \Omega \times (0,T)$, then the parabolic boundary of Q_T, denoted as $\partial_p Q_T$, is:

$$\partial_p Q_T = \left\{ (x_1, x_2, \cdots, x_d, t) \in Q_T \middle| (x_1, x_2, \cdots, x_d) \in \partial\Omega \text{ or } t = 0 \right\}.$$

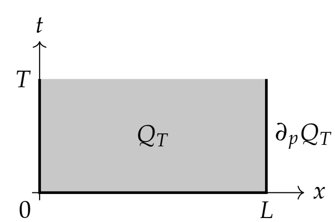

Figure 2.2. Parabolic Boundary for Region Q_T where Ω is the closed interval from 0 to L.

Therefore, we can discuss the respective equilibrium states of the heat equation.

Example 2.2.4. Equilibrium States for 1-D Rod.

For respective models, we can have the equilibrium states as:

(i) Dirichlet Boundary Condition: Assume the system as:

$$\begin{cases} \text{PDE:} & u_t = \kappa u_{xx}, \text{ where } x \in (0, L) \text{ and } t > 0; \\ \text{I.C.:} & u(x, 0) = f(x); \\ \text{B.C.:} & u(0, t) = T_1, \ u(L, t) = T_2. \end{cases}$$

Then, the equilibrium is:

$$u(x) = \frac{T_2 = T_1}{L} x + T_1,$$

which can be visualized as having a linear connection between the end points as $u'' = 0$.

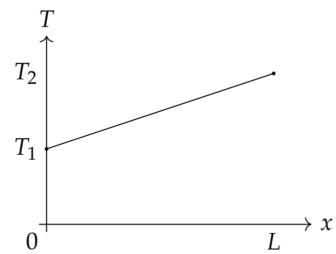

Figure 2.3. Equilibrium State for Dirichlet Boundary Condition.

(ii) Neumann Boundary Condition: Assume the system as:

$$\begin{cases} \text{PDE:} & u_t = \kappa u_{xx}, \text{ where } x \in (0, L) \text{ and } t > 0; \\ \text{I.C.:} & u(x, 0) = f(x); \\ \text{B.C.:} & u_x(0, t) = 0, \ u_x(L, t) = 0. \end{cases}$$

By the initial conditions and $u''(x) = 0$, we have $\tilde{u}(x) = c$, which is a constant. Thus, by the conservation of energy of the entire rod, we have:

$$\frac{d}{dt} \int_0^L c\rho u dx = -\kappa_0 u_x(0, t) + \kappa_0 u_x(L, t) = 0,$$

which implies that it must be a constant, hence by the initial conditions:

$$\int_0^L c\rho u(0, x) dx = \int_0^L c\rho f(x) dx,$$

and thus giving the equilibrium as:

$$u(x) = \lim_{t \to \infty} u(x, t) = \frac{1}{L} \int_0^L f(x) dx,$$

which is the average of the initial temperature distribution.

(iii) Non-homogeneous Example: Let the equation be:

$$\begin{cases} \text{PDE:} & u_t = u_{xx} + 1, \text{ where } x \in (0, L) \text{ and } t > 0; \\ \text{I.C.:} & u(x, 0) = f(x); \\ \text{B.C.:} & u_x(0, t) = 1, \ u_x(L, t) = \beta. \end{cases}$$

First, we want to use the limit to obtain that:

$$u''(x) + 1 = 1,$$

together with the boundary conditions to give that:

$$u(x) = -\frac{x^2}{2} + c_1 x + c_2, \text{ with } c_1 = 1 \text{ and } -L + 1 = \beta.$$

Consider the conservation of energy, we have:
$$\frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L u_{xx} + 1dx = u'(L,t) - u'(0,t) + L = \beta - 1 + L = 0.$$
Thus, we have for all $t \in \mathbb{R}^+$, the integral is a constant, which gives that:
$$\int_0^L f(x)dx = \int_0^L u(x,0)dx = \int_0^L u(x)dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right) dx,$$
providing us with:
$$c_2 = \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)dx,$$
so that:
$$u(x) = -\frac{x^2}{2} + x + \frac{L^2}{6} - \frac{1}{L}\int_0^L f(x)dx.$$

Remark 2.2.5. Boundary Conditions and Equilibrium in Other Case.

The boundary conditions and the equilibrium applies for extended cases other than 1-D rod:

(i) Global Cauchy problem: The global Cauchy problem describes the 1-D model on the whole $\mathbb{R}$. Its conditions are as follows:
$$\begin{cases} \text{PDE:} & u_t = \kappa u_{xx}, \text{ where } x \in \mathbb{R} \text{ and } t > 0; \\ \text{I.C.:} & u(x,0) = f(x), \text{ for } x \in \mathbb{R}; \\ \text{B.C.:} & \text{for } x \to \pm\infty. \end{cases}$$
Therefore, we have $|u(x,t)| \sim e^{-|x|}$, and in particular indicating that $\lim_{|x|\to\infty} |u(x,t)| \to 0$.

(ii) Higher dimensions: Suppose that we have a heat conducting body in a (open connected) bounded domain $\Omega \subset \mathbb{R}^d$ during the time interval $[0,T]$, the heat equation would be:
$$u_t(\mathbf{x},t) - \kappa\Delta u(\mathbf{x},t) = f(\mathbf{x},t)$$
in the space-time cylinder $Q_T = \Omega \times (0,T)$.
In this case, the boundary conditions is on $\partial\Omega \times (0,T]$ as:
$$\begin{cases} \text{Dirichlet:} & u = h; \\ \text{Neumann:} & \partial_{\mathbf{n}} u = h; \\ \text{Robin:} & \partial_{\mathbf{n}} + \alpha u = h, \ \alpha > 0. \end{cases}$$
Note that in the Neumann condition, we have the directional derivative as:
$$\partial_{\mathbf{n}} u = \nabla u \cdot \mathbf{n}.$$

Whether in 1-D or in higher dimensions, we want to define the parabolic boundary since we concern more about the boundary without the upper bound of the time variable.

2.3 Homogeneous Case, Method of Separation

Given the various boundary conditions for homogeneous cases, we can can be solving their solutions respectively.

Proposition 2.3.1. Principle of Superposition.

Let $\{u_n\}_{n=1}^{\infty}$ be a set of functions satisfying the PDE, then:

$$u_N(x,t) = \sum_{n=1}^{N} u_n$$

satisfies the PDE, and moreover:

$$u(x,t) = \lim_{N\to\infty} u_N(x,t) = \sum_{n=1}^{\infty} u_n$$

is a solution to the PDE.

The principal of Superposition allows the combinations of satisfactory solutions, whereas orthogonality allows us to reduct certain combinations.

Definition 2.3.2. Orthogonality.

For any *inner product space* V, with $\mathbf{u}, \mathbf{v} \in V$, we have:

$$\mathbf{u} \perp \mathbf{v} \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Specifically, for finite dimensional Euclidean space $\mathbb{R}^d$, for $\mathbf{u} = (u_1, u_2, \cdots, u_d), \mathbf{v} = (v_1, v_2, \cdots, v_d) \in \mathbb{R}^d$, we have orthogonality as:

$$\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{d} u_i v_i = 0.$$

On the other hand for the space of all continuous function on $[0, L]$, or $C[0, L]$, with $\varphi, \psi \in C[0, L]$, we have orthogonality as:

$$\varphi \perp \psi \iff \langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx = 0.$$

Orthogonality is defined base on an *inner product space*, specifically, we are interested in orthogonality in $C[0, L]$.

Corollary 2.3.3. Orthogonality for Sine and Cosine.

For non-negative integers m and n, the orthogonality of sine is:

$$\int_0^L \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} = \begin{cases} 0, & m \neq n \text{ or } m = n = 0; \\ L/2, & m = n \neq 0. \end{cases}$$

Likewise, the orthogonality of cosine is:

$$\int_0^L \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} = \begin{cases} 0, & m \neq n; \\ L/2, & m = n \neq 0; \\ L, & m = n = 0. \end{cases}$$

In the proving the *Orthogonality for Sine and Cosine*, we can apply trigonometric identity or the *Euler's Identity*.

Proposition 2.3.4. Euler's Identity.

Euler's identity concerns exponentials with imaginary powers.

$$e^{i\pi}+1=0 \qquad \text{or} \qquad e^{i\theta}=\cos\theta+i\sin\theta.$$

With *Euler's Identity*, some trigonometric identities can be derived, hence leading the proofs with orthogonality in $C[0,L]$.

Example 2.3.5. Solution to Homogeneous Heat Equation with Dirichlet Boundary Condition.

Given the following system of Heat equation:

$$\begin{cases} \text{PDE:} & u_t=\kappa u_{xx}, & \text{where } x\in(0,L) \text{ and } t>0; \\ \text{I.C.:} & u(x,0)=f(x), & \text{where } x\in[0,L]; \\ \text{B.C.:} & u(0,t)=0,\ u(L,t)=0, & \text{where } t\geq 0. \end{cases}$$

The idea is to convert the PDE to ODE, in which we let:

$$u(x,t)=\varphi(x)G(t),$$

which by assume that the ratios are the same, we convert the heat equation into:

$$\frac{G'(t)}{\kappa G(t)}=\frac{\varphi''(x)}{\varphi(x)}=-\lambda.$$

Thus, for all t and x, we convert each initial and boundary conditions of the heat equation into:

$$\begin{cases} \text{ODEs:} & G'(t)=-\lambda\kappa G(t), \\ & \varphi''(x)=-\lambda\varphi(x); \\ \text{I.C.:} & G(0)=f(x)/\varphi(x); \\ \text{B.C.:} & \varphi(0)=\varphi(L)=0. \end{cases}$$

When evaluating the first ODE, we have:

$$G(t)=G(0)e^{-\lambda\kappa t},$$

and for $G(0)>0$, if $\lambda<0$, then we have that $\lim_{t\to\infty}G(t)\to+\infty$, which is not possible, then $\lambda\geq 0$.
On the other hand evaluating the second ODE, we have $\varphi(x)=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$. Here, if $\lambda\leq 0$, we have $\varphi(x)\equiv 0$, which is the constant equation, while we are not interested in this case. Thus, we consider $\lambda>0$ for the system.
Applying the *Euler's formula*, we have:

$$\varphi(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x).$$

By implementing in the Boundary conditions, we have:

$$\begin{cases} \varphi(0)=C_1=0; \\ \varphi(L)=C_2\sin(\sqrt{\lambda}L)=0. \end{cases} \implies \begin{cases} C_1=0; \\ \sqrt{\lambda}L=n\pi \text{ for } n\in\mathbb{Z}^+. \end{cases}$$

Thus, the eigenvalues to $\mathcal{L}\varphi=\varphi''+\lambda\varphi=0$ are:

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2, \qquad \text{for } n=1,2,\cdots,$$

which applies to the eigenfunctions to $\varphi(0)=\varphi(L)=0$ as:

$$\varphi_n=\sin\left(\frac{n\pi}{L}x\right), \qquad \text{for } n=1,2,\cdots.$$

Thus, we can combine the separation of variables to get:

$$u_n(x,t) = \varphi_n(x)G_n(t) = B_n \sin(\sqrt{\lambda_n}x)e^{-\lambda_n \kappa t} = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-(n\pi/L)^2\kappa t}.$$

From here, by the *Principle of Superposition*, we may obtain that:

$$u_N = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi}{L}\right) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-(n\pi/L)^2\kappa t}$$

is a solution to the heat equation. More importantly, as $n \to \infty$, we have:

$$u(x,t) = \sum_{n=1}^{\infty} B_N \sin\left(\frac{n\pi}{L}x\right) e^{-(n\pi/L)^2\kappa t}$$

as the solution. Then, by applying the *Orthogonality*, we have:

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = \int_0^L \left[\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\right] \sin\left(\frac{m\pi}{L}x\right) dx = \int_0^L B_n \sin^2\left(\frac{m\pi}{L}x\right) dx = \frac{L}{2}B_m,$$

which gives us each B_m as:

$$B_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx.$$

From above, we proved the existence of a solution, then we want to investigate if the solution is unique.

2.4 Uniqueness and Maximum Principle

If we have uniqueness, the solution from *Method of Separation* serves as the existence of unique solution.

Definition 2.4.1. Uniqueness solution.

Let $u(x,t)$ and $v(x,t)$ be any solution to the PDE, then we have a unique solution if $u(x,t) = v(x,t)$.

Theorem 2.4.2. Uniqueness of Solutions to Homogeneous Heat Equations.

Let the Heat equation be:

$$\begin{cases} \text{PDE:} & u_t = \kappa\Delta u, \text{ where } x \in (0,L_1)\times(0,L_2)\times\cdots\times(0,L_d) \text{ and } t>0; \\ \text{I.C.:} & u(x,0) = f(x), \text{ where } x \in [0,L_1]\times[0,L_2]\times\cdots\times[0,L_d]; \\ \text{B.C.:} & \begin{cases} \text{(Dirichlet:)} & u = h, \\ \text{(Neumann:)} & \partial_{\mathbf{n}}u = h, \end{cases} \text{ where } t \geq 0. \end{cases}$$

The solution is unique.

Here, we first suppose that $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$ are two arbitrary solutions to the Heat equation, we want to construct $w(\mathbf{x},t)$ as:

$$w(\mathbf{x},t) := u(\mathbf{x},t) - v(\mathbf{x},t),$$

and we want to prove that it is constantly 0. By construction, we can use the Heat equation to get:

$$\begin{cases} \text{PDE:} & w_t - \kappa\Delta w = 0; \\ \text{I.C.:} & w(\mathbf{x},0) = 0; \\ \text{B.C.:} & \begin{cases} \text{(Dirichlet:)} & w = 0, \\ \text{(Neumann:)} & \partial_{\mathbf{n}}w = 0. \end{cases} \end{cases}$$

Then, we want to multiply the heat equation by w and integrate on Ω, where we find:

$$\int_\Omega w[w_t - \kappa\Delta w]d\mathbf{x} = 0 \Longrightarrow \int_\Omega ww_t d\mathbf{x} = \kappa\int_\Omega w\Delta w d\mathbf{x}.$$

Note that the left hand side is:

$$\int_\Omega ww_t d\mathbf{x} = \frac{1}{2}\frac{d}{dt}\int_\Omega w^2 d\mathbf{x},$$

and by *Green's first identity*, we have that:

$$\int_\Omega w\Delta w d\mathbf{x} = \int_{\partial\Omega} w\partial_{\mathbf{n}} w dS - \int_\Omega |\nabla w|^2 d\mathbf{x}.$$

Here, we define $E(t)$ as:

$$E(t) = \int_\Omega |w(\mathbf{x},t)|^2 d\mathbf{x},$$

which by taking the derivative with respect to t, we get:

$$\frac{1}{2}E'(t) = \int_\Omega ww_t dx = \kappa\int_\Omega w\Delta w d\mathbf{x} = \kappa\int_{\partial\Omega} w\partial_{\mathbf{n}} w dS - \kappa\int_\Omega |\nabla w|^2 d\mathbf{x}.$$

By the boundary conditions, we have:

$$\kappa\int_{\partial\Omega} w\partial_{\mathbf{n}} w dS = 0,$$

which simplifies $E'(t)$ into:

$$E'(t) = -\kappa\int_\Omega |\nabla w|^2 d\mathbf{x} \leq 0.$$

Hence, we know that $E(t) \to E(0) = \int_\Omega |0| d\mathbf{x} = 0$, however, since $E(t)$ is monotonically decreasing and non-negative, this implies that $E(t) = 0$, hence $w(\mathbf{x},t) \equiv 0$. Thus, the solution must be unique.

Theorem 2.4.3. Uniqueness in Solution for L^2 Space.

Let Ω be a bounded Lipschitz domain and $g \in L^2(\Omega)$ be a square integrable function in Ω. The initial Dirichlet, Neumann, Robin, and mixed problems have at most 1 solution, continuous in $\overline{\Omega} \times (0,T)$.
Moreover, the first and second spatial derivatives satisfies that:

$$\int_\Omega (w(\mathbf{x},t) - g(\mathbf{x}))^2 d\mathbf{x} \to 0 \text{ as } t \to 0^+.$$

Then, we the maximum principles would limit the existence of maximum temperature in a heat equation.

Definition 2.4.4. Subsolution and Supersolution.

For a function $\varphi \in C^{2,1}(Q_T)$ such that $\varphi_t - D\Delta\varphi \leq 0$ (or $\varphi_t - D\Delta\varphi \geq 0$, respectively) in Q_T is a subsolution (or supersolution) of the diffusion equation.

With such definition, we want to find the maximum of minimum value of a function in the closure of the given set. In particular, we want to find the location that the maximum and minimum temperature appears on the respective body.

Theorem 2.4.5. (Weak) Maximal Principle.

Let $\varphi \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ such that:

$$\varphi_t - D\Delta\varphi = q(\mathbf{x},t) \leq 0 \qquad (\text{or } \geq 0, \text{respectively}) \text{ in } Q_T,$$

then φ attains its maximum (or minimum) on $\partial_p Q_T$, *i.e.*:

$$\max_{\overline{Q_T}} \varphi = \max_{\partial_p Q_T} \varphi \qquad (\text{or } \min_{\overline{Q_T}} \varphi = \min_{\partial_p Q_T} \varphi).$$

In particular, if φ is negative (or positive, respectively) on $\partial_p Q_T$, then it is negative (or positive) in all $\overline{Q_T}$.

In proving the theorem, we want to consider two cases:

- Specific case. Assume that $\varphi \in C^{2,1}(\overline{Q_T})$ and that $q(\mathbf{x}, t) < 0$ for all $(\mathbf{x}, t) \in \overline{Q_T}$. For the sake of contradiction, suppose that φ attains its maximum at a point $(\mathbf{x}_0, t_0) \notin \partial_p Q_T$, then $x_0 \in \Omega((0, L))$ and $t_0 \in (0, T]$. By using calculus, we have that $\varphi_{x_j x_j}(\mathbf{x}_0, t_0) \leq 0$ for every $j = 1, 2, \cdots, n$, so we have:
$$\Delta\varphi(\mathbf{x}_0, t_0) = 0,$$
and thus:
$$\text{either } \varphi_t(\mathbf{x}_0, t_0) = 0 \text{ if } t_0 < T \text{ or } \varphi_t(\mathbf{x}_0, T) \geq 0,$$
which is a contradiction that:
$$0 \leq \varphi_t(\mathbf{x}_0, t_0) - D\varphi(\mathbf{x}_0, t_0) = q(\mathbf{x}_0, t_0) = 0.$$

- General case: Consider $\varphi \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, we can reduce to the previous case by letting $\epsilon \in (0, T)$ and $u = \varphi - \epsilon t$. Then:
$$u_t - D\Delta u = q - \epsilon < 0,$$
and $u \in C^{2,1}(\overline{Q_{T-\epsilon}})$, reducing us to:
$$\max_{\overline{Q_{T-\epsilon}}} \varphi \leq \max_{\overline{Q_{T-\epsilon}}} u + \epsilon T \leq \max_{\partial_p Q_T} u + \epsilon T \leq \max_{\partial_p Q_T} \leq \varphi + \epsilon T.$$
Since φ is continuous in $\overline{Q_T}$, we can reduct to that:
$$\max_{\overline{Q_{T-\epsilon}}} \varphi \to \max_{\overline{Q_T}} \varphi \text{ as } \epsilon \to 0.$$
Thus, we find that $\max_{\overline{Q_T}} \varphi \leq \max_{\partial_p Q_T} \varphi$, which concludes the proof.

Corollary 2.4.6. Maximal Principle in Homogeneous Case.

Let $w \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$. If:

$$w_t - D\Delta w = 0 \text{ in } Q_T,$$

then w attains its maximum and minimum on $\partial_P Q_T$, or *i.e.*:

$$\min_{\partial_p Q_T} w \leq w(\mathbf{x}, t) \leq \max_{\partial_p Q_T} w.$$

The above boundedness is an immediate result of *Weak Maximal principle*. Further, the theorem can be extended to comparison of multiple bounded function in Q_T.

Corollary 2.4.7. Comparison and Stability.

Let $v, w \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ be solutions to:

$$v_t - D\Delta v = f_1 \text{ and } w_t - D\Delta w = f_2,$$

where f_1 and f_2 are bounded in Q_T. Then:

(i) If $v \leq w$ on $\partial_P Q_T$ and $f_1 \geq f_2$ in Q_T, then $v \geq w$ in all $\overline{Q_T}$.

(ii) The following stability estimate holds:

$$\max_{\overline{Q_T}} |v - w| \leq \max_{\partial_p Q_T} |v - w| + T \cdot \sup_{Q_T} |f_1 - f_2|.$$

The first statement follows trivially. Let $u(\mathbf{x}, t) = v(\mathbf{x}, t) - w(\mathbf{x}, t)$, then we can get the following conditions on f:

$$\partial_t u - D\Delta u = \partial_t(v + w) - D\Delta(v + w) = (\partial_t v - D\Delta v) - (\partial_t w - D\Delta w) = f_1 - f_2 \geq 0.$$

Thereby, by the Maximum principal, as $\partial_t u - D\Delta u \geq 0$, this implies that:

$$\min_{\overline{Q_T}} u = \min_{\partial_p Q_T} u.$$

Moreover, from the condition that $v \geq w$ on $\partial_p Q_T$, we have $u = v - w \geq 0$ for $\partial_p Q_T$, implying that $\min_{\partial_p Q_T} u \geq 0$, and hence $\min_{\overline{Q_T}} u \geq 0$, implying that $v - w = u \geq 0$ for $\overline{Q_T}$, and $v \geq w$ for all $\overline{Q_T}$, satisfying the implication.

For the second implication, we let $u(\mathbf{x}, t) = v(\mathbf{x}, t) - w(\mathbf{x}, t)$. Moreover, we denote $M := \sup_{Q_T} |f_1 - f_2|$, then we can construct the following functions:

$$\begin{cases} Z_+(\mathbf{x}, t) = u(\mathbf{x}, t) - Mt, \\ Z_-(\mathbf{x}, t) = -u(\mathbf{x}, t) - Mt. \end{cases}$$

First, we shall first be studying Z_+, there, we have:

$$\begin{aligned} \partial_t Z_+ - D\Delta Z_+ &= \partial_t \left(u(\mathbf{x}, t) - Mt\right) - D\Delta \left(u(\mathbf{x}, t) - Mt\right) = \partial_t u - M - D\Delta u \\ &= \partial_t(v - w) - D\Delta(v - w) - M = \partial_t v - \partial_t w - D\Delta v + D\Delta w - M = f_1 - f_2 - M. \end{aligned}$$

Since by definition, $M = \sup_{Q_T} |f_1 - f_2|$, then within Q_T, we have $f_1 - f_2 \leq M$, hence:

$$\partial_t Z_+ - D\Delta Z_+ \leq 0,$$

which by the maximal principle and since M is non-negative, we have:

$$\max_{\overline{Q_T}} \big(u(\mathbf{x}, t) - Mt\big) = \max_{\overline{Q_T}} Z_+(\mathbf{x}, t) = \max_{\partial_p Q_T} Z_+(\mathbf{x}, t) = \max_{\partial_p Q_T} \big(u(\mathbf{x}, t) - Mt\big).$$

Since we have $\sup_{Q_T} |f_1 - f_2|$ being non-negative, and since we have $t \leq T$ for all t, then:

$$\begin{aligned} \max_{\overline{Q_T}} \big(u(\mathbf{x}, t) - MT\big) \leq \max_{\overline{Q_T}} \big(u(\mathbf{x}, t) - Mt\big) &= \max_{\partial_p Q_T} \big(u(\mathbf{x}, t) - Mt\big) \leq \max_{\partial_p Q_T} u(\mathbf{x}, t), \\ \max_{\overline{Q_T}} u(\mathbf{x}, t) - MT &\leq \max_{\partial_p Q_T} u(\mathbf{x}, t), \\ \max_{\overline{Q_T}} u(\mathbf{x}, t) &\leq \max_{\partial_p Q_T} u(\mathbf{x}, t) + MT, \\ \max_{\overline{Q_T}} (v - w) &\leq \max_{\partial_p Q_T} (v - w) + MT. \end{aligned}$$

For the Z_- case, it follows along in the same way with Z_+ with negative signs replacing signs, but the relations holds the same way. Therefore, combining the arguments with Z_+ and Z_-, we have:

$$\begin{cases} \max_{\overline{Q_T}}(v-w) \leq \max_{\partial_p Q_T}(v-w) + MT. \\ \max_{\overline{Q_T}}(w-v) \leq \max_{\partial_p Q_T}(w-v) + MT. \end{cases}$$

Since for any function f, and any set of input X, we have $\max_X |f| = \max\{\max_X(f), \max_X(-f)\}$. Thus, we have that:

$$\begin{aligned} \max_{\overline{Q_T}} |v-w| &= \max\left\{ \max_{\overline{Q_T}}(v-w), \max_{\overline{Q_T}}(w-v) \right\} \leq \max\left\{ \max_{\partial_p Q_T}(v-w) + MT, \max_{\partial_p Q_T}(w-v) + MT \right\} \\ &= \max\left\{ \max_{\partial_p Q_T}(v-w), \max_{\partial_p Q_T}(w-v) \right\} + MT = \max_{\partial_p Q_T} |v-w| + MT, \end{aligned}$$

as desired.

The *Weak Maximal principle* says nothing about the possibility that a solution achieves its maximum or minimum at an interior point. A more precise principle is as follows.

Theorem 2.4.8. Strong Maximal Principle.

Let $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ be a subsolution (or supersolution, respectively) of the heat equation in Q_T. If u attains its maximum (or minimum) M at a point $(\mathbf{x}_1, t_1)$ with $x_1 \in \Omega$ and $0 < t_1 \leq T$, then:

$$u \equiv M \text{ in } \overline{Q_{t_1}}.$$

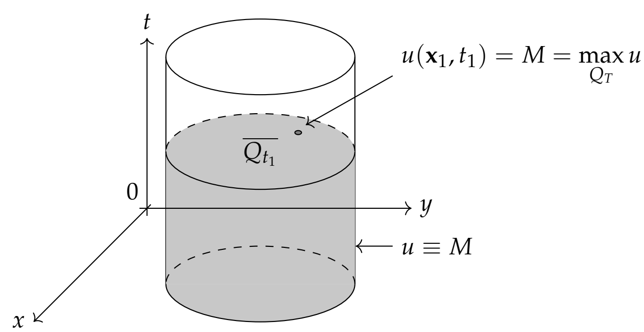

Figure 2.4. The Strong Maximal Principle.

2.5 Heat Kernels

In researching the Heat equations for global Cauchy problem in finite dimension d, we can find a model for solutions, called the heat kernel. In general, we want to solve the general Heat equation below:

$$\begin{cases} \text{PDE:} & u_t - D\Delta u = 0, \text{ where } x \in \mathbb{R}^d \text{ and } t \in (0,T); \\ \text{I.C.:} & u(x,0) = g(x), \text{ where } x \in \mathbb{R}^d; \\ \text{B.C.:} & \text{conditions as } |x| \to \infty. \end{cases}$$

Definition 2.5.1. Heat Kernel for the Global Cauchy Problem.

The *Heat kernel* or *Fundamental solution*, is:

$$\Phi_0(\mathbf{x}, t) = \frac{1}{\sqrt{(4\pi Dt)^d}} \exp\left(-\frac{|\mathbf{x}|^2}{4Dt} \right), \text{ for } t > 0, \mathbf{x} \in \mathbb{R}^d$$

Proposition 2.5.2. Properties of the Heat Kernel.

The *Heat Kernel* follows a series of properties:

(i) Integrability to 1: For $t > 0$, we have:
$$\int_{\mathbb{R}^n} \Phi_0(\mathbf{x}, t) d\mathbf{x} = 1;$$

(ii) Satisfying the PDE for Heat Equation: By taking derivatives, we have:
$$\partial_t \Phi_0(\mathbf{x}, t) = D\Delta\Phi_0(\mathbf{x}, t);$$

(iii) Point-wise Convergence: Let $\mathbf{x}$ be a fixed point, as $t \to 0^+$, $\Phi_0(\mathbf{x}, t)$ converges to:
$$\lim_{t\to 0^+} \Phi_0(\mathbf{x}, t) = 0 \text{ and } \lim_{t\to 0^+} \Phi_0(0, t) = +\infty,$$
which can be expressed as:
$$\Phi_0(\mathbf{x}, t) \to \delta(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \neq 0; \\ +\infty, & \mathbf{x} = 0. \end{cases}$$

With such properties, we can verify that the *heat kernel* is the solution to the *1-D Global Cauchy Problem*.

Corollary 2.5.3. Solution to 1-D Global Cauchy Problem.

The solution to the 1-D Global Heat Equation ($d = 1$) is:
$$u(x, t) = \int_{\mathbb{R}} \Phi_0(x - y, t) g(y) dy = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4Dt}\right) g(y) dy.$$

Note that we can also conclude the general solution to the bounded Heat equations.

Theorem 2.5.4. General Solution to 1-D Bounded Problem.

For Dirichlet Problem, the solution is:
$$u(x, t) = \sum_{n=1}^{\infty} B_n \exp\left(-\left(\frac{n\pi x}{L}\right)^2 \kappa t\right),$$
where the constants are:
$$B_n = \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \sin\left(\frac{n\pi x}{L}\right).$$
Meanwhile, for the Neumann Problem, the solution is:
$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\left(\frac{n\pi x}{L}\right)^2 \kappa t\right) \cos\left(\frac{n\pi x}{L}\right).$$
where the constants are:
$$A_0 = \frac{1}{L} \int_0^L f(x) dx,$$
$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Example 2.5.5. Homogeneous Heat Equation in Circular Ring.

In considering the case, we must be expanding the circular ring to a line segment.

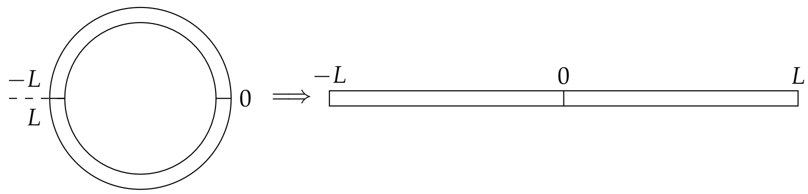

Figure 2.5. Expanding a circular ring to line segment for analysis.

Here, we can get the heat equations as:

$$\begin{cases} \text{PDE:} & u_t = \kappa u_{xx}, \text{ where } x \in (-L, L) \text{ and } t > 0; \\ \text{I.C.:} & u(x,0) = f(x), \text{ for } x \in [-L, L]; \\ \text{B.C.:} & u(-L,t) = u(L,t) = 0 \text{ and } u_x(-L,t) = u_x(L,t) = 0, \text{ for } t > 0. \end{cases}$$

The problem is a combination of two boundary conditions, hence the solution is:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-\left(\frac{n\pi x}{L}\right)^2 \kappa t\right) \cos\left(\frac{n\pi x}{L}\right) dx + \sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{n\pi x}{L}\right)^2 \kappa t\right) \sin\left(\frac{n\pi x}{L}\right) dx,$$

with coefficients being:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,$$
$$a_n = \frac{1}{2L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$
$$b_n = \frac{1}{2L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

2.6 Non-homogeneous Case, Duamel's Method

Duamel's Method accounts for the solving of non-homogeneous heat equations. The focus is to solve the following equation:

$$\begin{cases} \text{PDE:} & u_t - Du_{xx} = f(x,t), \text{ where } x \in \mathbb{R} \text{ and } t \in (0,T); \\ \text{I.C.:} & u(x,0) = g(x), \text{ where } x \in \mathbb{R}. \end{cases}$$

In this case, our strategy is to split the equation into:

$$u(x,t) = v(x,t) + w(x,t),$$

which respectively satisfies that:

$$\begin{cases} v_t - Dv_{xx} = 0, \\ v(x,0) = g(x). \end{cases} \quad \text{and} \quad \begin{cases} w_t - Dw_{xx} = f(x,t), \\ w(x,0) = 0. \end{cases}$$

Therefore, we can rewrite w as:

$$w(x,t) = \int_0^t h(x,t;s) ds,$$

which satisfies that:

$$\begin{cases} h_t - Dh_{xx} = 0, \\ h(x,s;s) = f(x,s). \end{cases}$$

2.7 2-D Heat Equation on $[0,a] \times [0,b]$

In general, we can also apply the method of separation for higher dimensions as well.

Example 2.7.1. 2-D Heat Equation.

Consider the Heat equation on the rectangle $[0,a] \times [0,b]$ with Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} & (x,y) \in (0,a) \times (0,b), t > 0; \\ u(x,y,0) = g(x,y) & (x,y) \in [0,a] \times [0,b]; \\ u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0; & 0 < y < b, t > 0. \end{cases}$$

By the method of separation, we let:

$$u(x,y,t) = \phi(t)p(x,y) = \phi(t)v(x)w(y).$$

By the first separation, we have:

$$\phi'(t)p(x,y) = \phi(t)(\Delta p(x,y)) \Longrightarrow \frac{\phi'(t)}{\phi(t)} = \frac{\Delta p(x,y)}{p(x,y)} = -\lambda.$$

By the second separation, we have:

$$v''(x)w(y) + v(x)w''(y) = -\lambda v(x)w(y) \Longrightarrow \frac{v''(x)}{v(x)} = -\lambda - \frac{w''(x)}{w(x)} = -\mu,$$

with the boundary conditions transferred into:

$$v(0) = v(a) = w(0) = w(b) = 0.$$

With these, we first solve for $v(x)$, with:

$$\begin{cases} v''(x) = -\mu v(x), \\ v(0) = v(a) = 0. \end{cases}$$

Consider if $\mu \leq 0$, this implies that the boundary conditions give trivial solutions. For $\mu > 0$, we have:

$$v(x) = A\sin(\sqrt{\mu}x) + B\cos(\sqrt{\mu}x),$$

and by boundary conditions, we have $B = 0$ and $\sqrt{\mu_m} = \frac{m\pi}{a}$.
Then, considering $w(y)$, we have likewise that for $\lambda - \mu > 0$ so we have:

$$w(x) = C\sin(\sqrt{\lambda - \mu}x) + D\cos(\sqrt{\lambda = \mu}x),$$

in which we have $D = 0$ and $\sqrt{\lambda - \mu} = \frac{n\pi}{b}$, so we have the general solution as:

$$p_{mn}(x,y) = a_{mn}\sin\frac{n\pi y}{b}\sin\frac{m\pi x}{a},$$

which we have $\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$, hence, we solve ϕ as:

$$\phi(t) = e^{-\lambda_{mn}t},$$

giving us the general solution that:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\lambda_{mn}t}\left(a_m n \sin\frac{n\pi y}{b}\sin\frac{m\pi x}{a}\right),$$

and by the initial conditions, we have:

$$a_{mn} = \frac{2}{a}\int_0^a \frac{2}{b}\int_0^b g(x,y)\sin\frac{n\pi y}{b}dy\sin\frac{m\pi x}{a}dx.$$

3 Fourier Series

3.1 Fourier Transformation

Definition 3.1.1. 1-D Fourier Transformation.

Let f be a complex-valued function, the Fourier Transformation, denoted $\mathcal{F}$, is defined as:

$$\mathcal{F}[f](\xi) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(x)e^{i\xi x}dx.$$

Proposition 3.1.2. Properties of Fourier Transformation.

Let $f(x), h(x) \in L^1(\mathbb{C})$ and $a, b \in \mathbb{C}$, the *Fourier transformation* has the following properties:

(i) Linearity: $\mathcal{F}[af + bh](\xi) = a\mathcal{F}[f](\xi) + b\mathcal{F}[h](\xi)$;

(ii) Second Derivative Property: $\mathcal{F}[f''](\xi) = -\xi^2\mathcal{F}[f](\xi)$;

(iii) Periodicity: The Fourier transformation follows the periodicity that

$$\mathcal{F}^1[f](x) = \mathcal{F}[f](x), \qquad \mathcal{F}^2[f](x) = f(-x), \qquad \mathcal{F}^3[f](x) = \mathcal{F}^{-1}[f](x), \qquad \mathcal{F}^4[f](x) = f(x).$$

The properties of *Fourier transformation* also amounts to the properties of the *Heat equation*.

Proposition 3.1.3. Fourier Transformation in Heat Equation.

Let $u(x,t)$ be solution to heat equation, we have:

$$\partial_t\mathcal{F}[u(x,t)](\xi,t) - D\mathcal{F}[\partial_{xx}u(x,t)](\xi,t) = \mathcal{F}[u(x,t)](\xi,t).$$

Hence, this leads to the ODE that:

$$\partial_t H(t) + D\xi^2 H(t) = F(t),$$

where:

$$H(t) = \mathcal{F}[u(x,t)](\xi,t), \qquad F(t) = \mathcal{F}[f](\xi,t).$$

3.2 Fourier Series on $[-L, L]$

The canonical *Fourier series* expands with both sines and cosines, while the other cases are special cases derived from the more general case.

Definition 3.2.1. General Fourier Series on $[-L, L]$.

Let f be a real valued function that is integrable on $[-L, L]$. With the periodic boundary conditions being on $-L \leq x \leq L$, the *Fourier series* is an infinite series:

$$f(x) \rightsquigarrow a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficient are:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

To investigate whether the expansion is valid, we want to inquiry the following questions:

(i) Does the infinite series converge?

(ii) Does the infinite series converge to $f(x)$?

(iii) Is the series a solution of some PDEs?

Starting the investigation of the first question, we need to define a new class of functions.

Definition 3.2.2. Piecewise Smooth Functions.

A function $f(x)$ is *piecewise smooth* on some interval (a, b) if there exists a finite number of intervals $\{I_k\}_{k=1}^{n}$ with $I_k = (a_k, b_k]$ or $I_k = (a_k, b_k)$ that covers the open interval (a, b), *i.e.*, $(a, b) = \bigcup_{k=1}^{n} I_k$, such that in each piece I_k, $f\big|_{I_k}$ and $f'\big|_{I_k}$ are continuous and bounded.
The definition of smoothness is a stronger statement than purely continuity, which the following examples would demonstrate.

Remark 3.2.3. Examples of Piecewise Smooth Functions.

Finitely many jump discontinuity does not necessarily violate the piecewise smoothness, while continuity does guarantee piecewise smoothness.

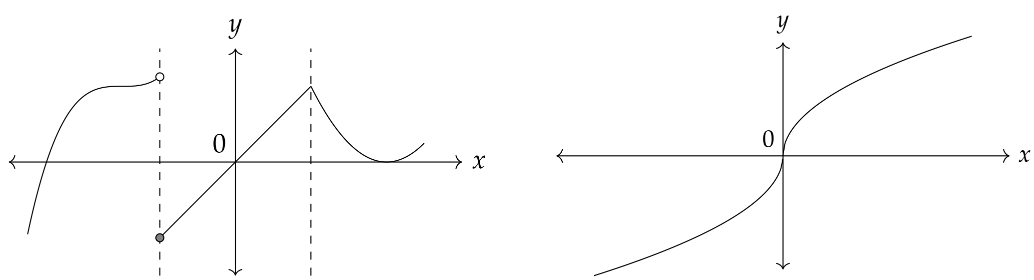

Figure 3.1. The left function is piecewise smooth but the right function is not (as its derivative not bounded).

In general, the idea of *Fourier series* is to have a period extension of $f(x)$, the following example extends the periodicity of a function.

Example 3.2.4. Periodic Expansion of x^2.

Let $f(x) = x^2$. The *Periodic expansion* of $f(x)$ expands the function with every $2L$ as an interval.

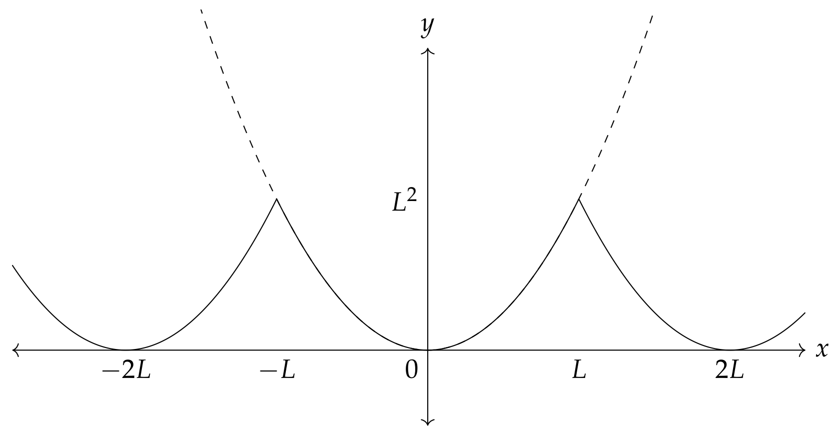

Figure 3.2. Periodic expansion of $f(x) = x^2$ has expanded periodicity of $2L$.

With the above example, we do want to investigate on what functions would allow the *Fourier series* to converge, leading to the convergence theorem.

Theorem 3.2.5. Convergence Theorem for Fourier Series.

If $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$, then the Fourier series of $f(x)$ is:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficient are:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Moreover, the *Fourier series* of $f(x)$ converges. Furthermore, the series converges to:

(i) the periodic extension of $f(x)$, where the periodic extension is continuous;

(ii) the average of the two limits, usually $(f(x+) + f(x-))/2$, where the periodic extension has a *jump discontinuity*.

In proving the theorem, we consider without the loss of generality that $L = \pi$ and $-\pi \leq x \leq \pi$.

- In the first case, we assume that f is smooth on $[-\pi, \pi]$, in which we have:

$$S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)),$$

  with the coefficients being:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

  Therefore, our *Fourier series* becomes:

$$\begin{aligned} S_N(x) &= \int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{N}[\cos(ny)\cos(nx) + \sin(ny)\sin(nx)]\right] f(y)\frac{dy}{2\pi} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{N}\cos\left(n(x-y)\right)\right] f(y)dy. \end{aligned}$$

Proposition 3.2.6. Finite Sum of a Trigonometric Function.

We claim that:

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos(n\theta) = \frac{\sin(N+1/2)\theta}{\sin(\theta/2)}, \text{ and } \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1.$$

The prove of the claim is trivial through *Euler's identity*, that is:

$$\begin{aligned} K_N(\theta) &= 1 + \sum_{n=1}^{N}(e^{in\theta} + e^{-in\theta}) = e^{-i(N+1/2)\theta} + \cdots + 1 + \cdots e^{i(N+1/2)\theta} \\ &= \frac{e^{-N\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} = \frac{e^{-i(N+1/2)\theta} - e^{i(N+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}} = \frac{\sin[(N+1/2)\theta]}{\sin(\theta/2)}. \end{aligned}$$

The second claim follows directly since $\int_{-\pi}^{\pi} \cos(n\theta) = 0$.
Then, we can write S_N as:

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y)f(y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)f(x+\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)f(x-\theta)d\theta.$$

Since $f(x)$ and $K_N(x)$ are 2π-periodic function, so:

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)[f(x+\theta) - f(\theta)]d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x+\theta) - f(\theta)}{\sin(\theta/2)} \sin[(N+1/2)\theta]d\theta.$$

Note that the function $g(\theta) := \dfrac{f(x+\theta) - f(\theta)}{\sin(\theta/2)}$ is piecewise continuous on $[a, b]$, then by the *Riemann-Lebesgue Lemma*, we have:

$$g(\theta) = \frac{f(x+\theta) - f(x)}{\sin(\theta/2)} = \frac{f(x+\theta) - f(x)}{\theta} \cdot \frac{\theta}{\sin(\theta/2)},$$

and thus giving us that:

$$\begin{aligned} \left|\int_{-\pi}^{\pi} g(x)\sin((N+1/2)\theta)d\theta\right| &= \left|\frac{-1}{N+1/2}\int_{-\pi}^{\pi} g(\theta)d\cos((N+1/2)\theta)\right| \\ &\leq \frac{1}{N+1/2}\left|g(\pi)\cos((N+1/2)\theta\pi) - g(-\pi)\cos(-(N+1/2)\theta\pi)\right| \\ &\quad + \frac{1}{N+1/2}\left|\int_{-\pi}^{\pi} \cos((N+1/2)\theta)g'(\theta)d\theta\right| \\ &\leq \frac{M_1}{N+1/2} + \frac{M_2 \cdot 2\pi}{N+1/2} \to 0 \text{ as } N+1/2 \to \infty. \end{aligned}$$

Thus, we have concluded that for the piecewise smooth function f, we have the *Fourier series* converging.

- For the second case, we want to consider when f is not continuous on at some $x \in [-\pi, \pi]$.
  Here, we want to look at:

$$\begin{aligned} S_N(x) - \frac{1}{2}[f(x+) + f(x-)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)f(x+\theta)d\theta - \frac{1}{2}[f(x+) + f(x-)] \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} K_N(\theta)[f(x+) - f(x-)]d\theta \\ &\quad + \frac{1}{2\pi}\int_{0}^{\pi} K_N(\theta)[f(x+) - f(x-)]d\theta \end{aligned}$$

Considering one of the component, we can define:

$$g_+(\theta) = \begin{cases} 0, & -\pi \le \theta \le 0; \\ \dfrac{f(x+\theta) - f(x+)}{\theta} \cdot \dfrac{\theta}{\sin(\theta/2)}, & 0 < \theta \le \pi. \end{cases}$$

Since at $\theta \to 0^+$, we have $g(\theta) \to \frac{f(x+\theta)-f(x+)}{\theta} \cdot \frac{\theta}{\sin(\theta/2)} = 2f'(x+)$, so it is piecewise smooth. With the same logic of *Riemann Lebesgue Lemma*, we have both integrands disappearing and hence proving the pointwise convergence.

Remark 3.2.7. Comparing Fourier Series with Weierstrass Approximation Theorem.

The *Fourier series* allows only pointwise smoothness, which is different from the *continuity* in *Weierstrass approximation theorem*, while their approximates series differs that the *Fourier series* converges pointwise, but *Weierstrass approximation theorem* converges uniformly.

Without the stronger statement of uniform convergence, we want to investigate the accuracy of the approximation, which leads to the *Gibbs* Phenomenon.

Proposition 3.2.8. Gibbs Phenomenon.

The *Fourier series* does not necessarily converge uniformly. At the jump discontinuities, the *overshoot* is about 9%.

Here, we let $f(x)$ be the indicator function such that:

$$f(x) = \begin{cases} 0, & -\pi \le x < 0; \\ 1, & 0 \le x \le \pi. \end{cases}$$

Its *Fourier series* is:

$$F(x) = \lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \left[\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin(2n-1)x}{2n-1} \right],$$

and we hereby define $g_N(x)$ on $[0, \pi]$:

$$g_N(x) = S_N(x) - 1 = \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin(2n-1)x}{2n-1} - \frac{1}{2}.$$

Here, as we construct a series $\{x_N\}_{N=1}^{\infty}$ by $x_N = \frac{\pi}{2N}$. Meanwhile, as we notice that:

$$\int_0^{\pi} \frac{\sin\theta}{\theta} d\theta = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2n}{(2i-1)\pi} \cdot \sin\left[\frac{(2i-1)\pi}{2n}\right] \cdot \frac{\pi}{n} = \lim_{n\to\infty} 2 \sum_{i=1}^{n} \frac{1}{2i-1} \cdot \sin\left[\frac{(2i-1)\pi}{2n}\right],$$

we may apply it to the proof, then, we obtain that:

$$\begin{aligned} \lim_{N\to\infty} g_N(x_N) &= \frac{1}{\pi} \int_0^{\pi} \frac{\sin(x)}{x} dx - \frac{1}{2} = \frac{1}{\pi} \int_0^{\pi} (1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots) dx - \frac{1}{2} \\ &\approx 0.5 - 0.5483 + 0.1623 - 0.027 \approx 0.087. \end{aligned}$$

3.3 Uniform Convergence in Fourier Series

Still, *uniform convergence* is a more desirable results, so we want to investigate the conditions leading to uniform convergence. This involves in the family of function in $L^2([-L, L])$.

Theorem 3.3.1. L^2 **Convergence Theorem.**

Let $f \in L^2([-L, L])$ be a square integrable function, that is:

$$\int_{-L}^{L} |f(x)|^2 dx < +\infty,$$

then:

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 dx = 0.$$

Moreover, the *Parseval Identity* holds:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} [a_n^2 + b_n^2].$$

The proof of the L^2 *Convergence Theorem* involves more contents in L^2 space. The follows is an immediate result of the L^2 *Convergence Theorem*.

Corollary 3.3.2. Uniform Convergence Condition.

Let $f(x)$ and its derivative $f'(x)$ be continuous functions of period 2π, then:

$$\lim \sup_{x\in[-\pi,\pi]} |S_N(x) - f(x)| = 0.$$

Here, we let a_n, b_n be the Fourier coefficients of $f(x)$ and let A_n, B_n be the Fourier coefficients of $f'(x)$. By the definition of a_n, we apply integration by parts to have:

$$\begin{aligned} a_n &= \int_{-\pi}^{\pi} f(x)\cos(nx)\frac{dx}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ &= \frac{1}{\pi}\left[\frac{f(x)\sin(nx)}{nx}\right]_{x=-\pi}^{x=\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f'(x)\sin(nx)}{n}dx \\ &= 0 - \frac{1}{n}\int_{\pi}^{\pi} f'(x)\sin(nx)\frac{dx}{\pi} = -\frac{1}{n}B_n. \end{aligned}$$

By the definition of b_n, we apply integration by parts to have:

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} f(x)\sin(nx)\frac{dx}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\ &= \frac{1}{\pi}\left[-\frac{f(x)\cos(nx)}{nx}\right]_{x=-\pi}^{x=\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} -\frac{f'(x)\cos(nx)}{n}dx \\ &= 0 + \frac{1}{n}\int_{\pi}^{\pi} f'(x)\cos(nx)\frac{dx}{\pi} = \frac{1}{n}A_n. \end{aligned}$$

Give then $f'(x)$ is continuous on $[-\pi, \pi]$, and given that it is (uniformly) continuous and bounded, it belong to L^2. Hence, by the Parseval identity, we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = 2A_0^2 + \sum_{n=1}^{\infty} [A_n^2 + B_n^2] < +\infty.$$

Then, if we consider the metric between $S_N(x)$ and $f(x)$, we have:

$$\begin{aligned}
\sup_{x\in[-\pi,\pi]} |S_N(x) - f(x)| &= \sup_{x\in[-\pi,\pi]} \left| \sum_{n=N+1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right| \\
&\leq \sum_{n=N+1}^{\infty} [|a_n| + |b_n|] \\
&= \sum_{n=N+1}^{\infty} \left[\frac{1}{n}|B_n| + \frac{1}{n}|A_n| \right] \\
&\leq \frac{1}{2} \sum_{n=N+1}^{\infty} \left[\frac{1}{n} + B_N^2 + \frac{1}{n} + A_N^2 \right] \\
&= \sum_{n=N+1}^{\infty} \frac{1}{n^2} + \frac{1}{2} \sum_{n=N+1}^{\infty} [A_N^2 + B_N^2].
\end{aligned}$$

Since $\sum_{n=1}^{\infty}[A_n^2 + B + n^2]$ is convergent, this implies that $A_N^2 + B_N^2 < \epsilon$ for all some $\epsilon > 0$ for all $n > N$ given some N large enough.
Therefore, as $N \to \infty$, we have both terms vanishes, hence:

$$\lim \sup_{x\in[-\pi,\pi]} |S_N(x) - f(x)| \to 0.$$

Therefore, by the *Fourier theorem*, the functions converge uniformly for when itself and its derivate is continuous functions on a period of 2π.

3.4 Odd and Even Expansions

Then, we want to expand to the specific cases of *Fourier series*, which is the odd and even expansions.

Definition 3.4.1. Fourier Odd Expansion.

Let $f(x)$ be an odd function, *i.e.*, $f(-x) = -f(x)$, then we have:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients are:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right).$$

Definition 3.4.2. Fourier Even Expansion.

Let $f(x)$ be an even function, *i.e.*, $f(-x) = f(x)$, then we have:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where the coefficients are:

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

The *Fourier Odd* and *Fourier Even* expansions are more suitable to the odd and even function, for the unsuitable case, the expansion is more accurate on $(0, L)$, but not the other interval.

Example 3.4.3. Signed Fourier Expansion for $f(x) = x^2 + x$.

$f(x) = x^2 + x$ is a non-odd and non-even function, therefore, its odd and even expansion would not correspond to the whole interval $[-L, L]$. Account for expanding $(0, L)$, we can consider the cases for the *Signed Fourier expansion* for $f(x)$, respectively, are:

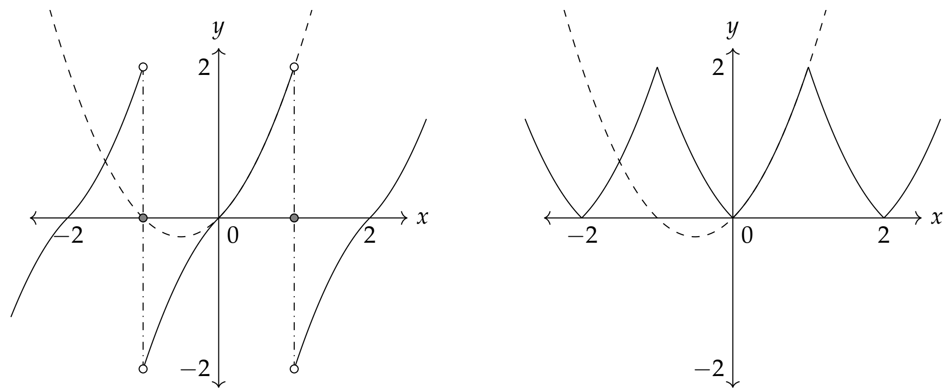

Figure 3.3. Odd (left) and Even (right) expansion of $f(x) = x^2 + x$.

3.5 Term-by-term Differentiation and Integration

The Fourier series serve as a model for the family of piecewise smooth function. Then, we also want to investigate if term-by-term differentiation and integration works for the family of functions.

Example 3.5.1. Counterexample for Term-by-term Differentiation.

Consider the Fourier sine series of x on the interval $0 \leq x \leq L$, then we have:

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right).$$

A term-by-term differentiation is that:

$$2\sum_{n=1}^{\infty}(-1)^{n+1} \cos\left(\frac{n\pi x}{L}\right).$$

Note that the Fourier cosine series is $f(x) = 1$, and moreover, the approximation does not converge to 1 almost everywhere. Hence, the differentiation term-by-term is not justified for the family of piecewise smooth function.

Now, we have proven (by counterexample) that the term-by-term differentiation does not apply for the whole family of piecewise smooth function. Now, we want to find the family in which term-by-term differentiation (and integration, respectively) holds.

Theorem 3.5.2. Term-by-term Differentiation for Fourier Series.

If $f(x)$ is piecewise smooth, then the *Fourier series* of a continuous function $f(x)$ can be differentiated term-by-term if $f(-L) = f(L)$.
Equivalent, a *Fourier series* that is continuous can be differentiated term-by-term if $f'(x)$ is piecewise smooth.

The alternatives holds equivalently since they both address the condition for the *Fourier series* to be continuous.

Then, we are going to consider the more specialized cases, *i.e.*, the cosine and sine expansion.

Corollary 3.5.3. Term-by-term Differentiation for Fourier Cosine Series.

If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term-by-term.

The term-by-term differentiation for Fourier cosine series works in the same way with the general series, but with a weaker condition as the equal condition is inherently achieved by symmetry. Diligent readers should already expect a stronger condition for Fourier sine series as the symmetry does not help with the equality but makes the condition more strict.

Corollary 3.5.4. Term-by-term Differentiation for Fourier Sine Series.

Let $f'(x)$ is piecewise smooth, the Fourier sine series of a continuous function $f(x)$ can be differentiated term-by-term if $f(0) = f(L) = 0$.

In certain cases, the term-by-term differentiation for Fourier Sine Series is not justified, however, we can compensate by giving a correction to the differentiation.

Proposition 3.5.5. Correction for Term-by-term Differentiation for Fourier Sine Series.

Let $f(x)$ be a piecewise smooth function that does not satisfy the condition for term-by-term differentiation. Let the Fourier sine series of $f(x)$ be:

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

then the derivative $f'(x)$ is corrected as follows:

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} B_n + \frac{2}{L}\big((-1)^n f(L) - f(0)\big)\right] \cos\left(\frac{n\pi x}{L}\right).$$

With the corrected formula, we may reconsider the previous counterexample.

Example 3.5.6. Correction to Counterexample for Fourier Sine Series.

For the Fourier sine series of x on the interval $0 \leq x \leq L$, then we have:

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right).$$

A corrected term-by-term differentiation is that:

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} \cdot \frac{2L}{n\pi}(-1)^{n+1} + \frac{2}{L}\big((-1)^n f(L) - f(0)\big)\right] \cos\left(\frac{n\pi x}{L}\right),$$

which simplifies to:

$$f'(x) \sim \frac{1}{L}(L - 0) + \sum_{n=1}^{\infty} [2(-1)^{n+1} + 2(-1)^n] \cos\left(\frac{n\pi x}{L}\right) = 1,$$

which corresponds to that $f'(x) = 1$. Thus, the correction to the term-by-term differentiation applies.

With the properties of term-by-term differentiations, we may also apply the properties to functions of multiple variables.

Corollary 3.5.7. Fourier Expansions to Multi-variable Functions.

Let $u(x,t)$ be a continuous function depending on a parameter t. Then *Fourier series* is:

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left[a_n(t) \cos\left(\frac{n\pi x}{L}\right) + b_n(t) \sin\left(\frac{n\pi x}{L}\right) \right].$$

If $\partial_t u(x,t)$ is piecewise smooth, it can be differentiated term-by-term with respect to t, yielding that:

$$\partial_t u(x,t) \sim a_0'(t) + \sum_{n=1}^{\infty} \left[a_n'(t) \cos\left(\frac{n\pi x}{L}\right) + b_n'(t) \sin\left(\frac{n\pi x}{L}\right) \right].$$

Such results can be immediately applied to the solution of the Heat equation with zero boundary conditions at $x = 0$ and $x = L$.

Example 3.5.8. Revisit the Solution to Heat Equation.

Here, we use the *Method of Eigenfunction Expansion* to solve the heat equation with zero boundary conditions at $x = 0$ and $x = L$. We suggest a *Fourier sine series* for each time, respectively:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

Hence, the initial condition is satisfied if:

$$f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin\left(\frac{n\pi x}{L}\right).$$

Then, we determine the coefficients of the Fourier sine series, which is:

$$B_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Afterwards, since the function is piecewise smooth and has the zero boundary conditions, differentiation term-by-term is justified. Hence, our goal is to solve for our candidate function that satisfies the heat equation ($u_t = \kappa u_{xx}$), which we calculate:

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n'(t) \sin\left(\frac{n\pi x}{L}\right),$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} -\frac{L^2}{(n\pi)^2} B_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

which gives us the relationship that:

$$\frac{dB_n(t)}{dt} = -\kappa \left(\frac{n\pi}{L}\right)^2 B_n(t),$$

and solving the ODE gives us that:

$$B_n(t) = B_n(0) e^{-(n\pi/L)^2 \kappa t}.$$

The term-by-term differentiation facilitates the solving of many PDEs, while on the other hand, the term-by-term integration, as the parallel to differentiation, would also be a handy tool for solving PDEs.

Theorem 3.5.9. Term-by-term Integration for Fourier Series.

A *Fourier series* of a piecewise smooth $f(x)$ can always be integrated term-by-term.
Furthermore, the result is a convergent infinite series that always converges to the integral of $f(x)$ for

$-L \leq x \leq L$ (even if the original Fourier series has jump discontinuities).

To verify the statement, let $f(x)$ be a piecewise smooth function and its Fourier series on $[-L, L]$ be:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

The term-by-term integration yields us with:

$$\begin{aligned}\int_{-L}^{x} f(t)dt &\sim a_0(x+L) + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{x} \cos\left(\frac{n\pi t}{L}\right) dt + b_n \int_{-L}^{x} \sin\left(\frac{n\pi t}{L}\right) dt\right] \\ &\sim a_0(x+L) + \sum_{n=1}^{\infty}\left[\frac{a_n}{n\pi/L} \sin\left(\frac{n\pi x}{L}\right) + \frac{b_n}{n\pi/L}\left((-1)^n - \cos\left(\frac{n\pi x}{L}\right)\right)\right].\end{aligned}$$

Thus, the term-by-term integration is justified. Moreover, we note a few more properties.

Remark 3.5.10. Note on Integration Fourier Series.

The term-by-term integration on Fourier Series follows the below properties:

(i) The term-by-term integration is justified for Fourier signed series;

(ii) The term-by-term integrated series is continuous;

(iii) The term-by-term integrated series may not be a Fourier series.

3.6 Fourier Series in Complex Form

Despite the usefulness of *Fourier series*, mathematicians always pursue the elegance and delicacy of formulation through *Euler's identity*.

Definition 3.6.1. Fourier Series in Complex Form.

Let $f(x)$ be a piecewise smooth function, its *Fourier series* in complex form is:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L},$$

where the respective coefficients are:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L} dx.$$

The derivation is purely the application of *Euler's identity* to formulate the information of cosine and sine expansions into a single complex number. At the same time, the construction of complex form also give rise to some properties with the coefficients.

Proposition 3.6.2. Properties on Coefficients of Complex Fourier Expansion.

Let $f(x)$ be a real-valued function, then $c_{-n} = \overline{c_n}$, where $\overline{c_n}$ is the complex conjugate of c_n.

Conceptually, the proposition is related to the odd and even natural of cosine and sine expansions, hence providing us with different results on how real (related to cosine) and imaginary (related to sine) parts corresponds to each other.

4 Wave Equations

4.1 Wave Equation on 1-D String

Different from the derivation of Heat equation, our wave equation follows more hypothesis when constructed.

Remark 4.1.1. Hypothesis to Wave Model.

The wave model obeys the following hypothesis:

(i) There are vertical displacement only, or the displacement must be orthogonal to the string (or plane);

(ii) The displacement $u(x,t)$ only depends on t and x;

(iii) The vibrations od string are small amplitude compared to its length, or $|v_x(x,t)| \ll 1$;

(iv) The string is perfectly flexible;

(v) Friction is negligible.

With the above assumptions, the equation of motion of the string can be derived using conservation of mass and Newton's law.

- Here, we first consider the *conservation of mass*.

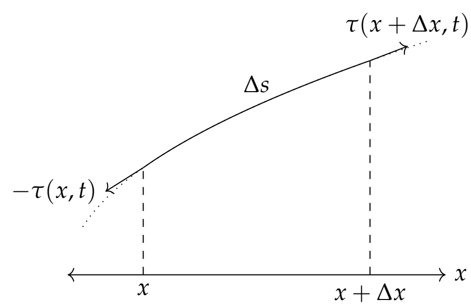

Figure 4.1. Tension force at boundaries of a small segment of string with vertical movements only.

Note that the string has been stretched. Let $\rho_0(x)$ be the linear density of the string at rest and $\rho(x,t)$ be its density at time t. Then the conservation of mass yields:

$$\rho_0(x)\Delta x = \rho(x,t)\Delta s.$$

Moreover, as the string has no horizontal movements, then let $\tau(x,t)$ be the magnitude of the tension at x at time t and let $\alpha(x,t)$ denote the angle between the tangent line with the horizontal, then:

$$\tau(x+\Delta x,t)\cos\alpha(x+\Delta x,t) - \tau(x,t)\cos\alpha(x,t) = 0,$$

By dividing Δx and letting $\Delta \to 0$, we have:

$$\lim_{\Delta x\to 0} \frac{\tau(x+\Delta x,t)\cos\alpha(x+\Delta x,t) - \tau(x,t)\cos\alpha(x,t)}{\Delta x} = \partial_x[\tau(x,t)\cos\alpha(x,t)] = 0,$$

thus implying that $\tau(x,t)$ is constant, which is:

$$\tau(x,t)\cos\alpha(x,t) = \tau_0(t) \text{ and } \tau_0(t) \text{ is positive.}$$

Then, we consider about the vertical component τ_{vert}, that satisfies with:

$$\tau_{\text{vert}}(x,t) = \tau(x,t)\sin\alpha(x,t) = \tau_0(t)\tan\alpha(x,t) = \tau_0(t)u_x(x,t).$$

Therefore, for the small segment, we have:

$$\tau_{\text{vert}}(x+\Delta x,t) - \tau_{\text{vert}}(x,t) = \tau_0(t)[u_x(x+\Delta x,t) - u_x(x,t)].$$

- Then, we utilize the *Newton's second law* of motion.
  Let $f(x,t)$ be the magnitude of the vertical body force per unit mass, then we have the magnitude of the body force at each string segment as:
  $$\int_x^{x+\Delta x} \rho(y,t)f(y,t)ds = \int_x^{x+\Delta x} \rho_0(y)f(y,t)dy.$$
  Since we concern about the vertical displacement, then the vertical u_{xx} can be the acceleration, that is the force per unit mass, then:
  $$\int_x^{x+\Delta x} \rho_0(y)u_{tt}(y,t)dy = \tau_0(t)[u_x(x+\Delta x,t) - u_x(x,t) - u_x(x,t)] + \int_x^{x+\Delta x} \rho_0(y)f(y,t)dy.$$
  When we divide by Δx again and let $\Delta x \to 0$, we have:
  $$u_{tt}(x,t) = \frac{\tau_0(t)}{\rho_0(x)}u_{xx}(x,t) + f(x,t).$$

By some simplification, we have achieved our formula for *wave equation*.

Theorem 4.1.2. Wave Equation on 1-D String.

Let $u(x,t)$ be the displacement of the string at position x and time t, we have:

$$u_{tt}(x,t) - c^2(x,t)u_{xx}(x,t) = f(x,t),$$

where $c(x,t) = \sqrt{\dfrac{\tau_0(t)}{\rho_0(x)}}$, and $f(x,t)$ is the external force.

Throughout the derivation, we note the formula for some special cases.

Remark 4.1.3. Constant Functions in Wave Equation on 1-D String.

With some specialized properties, we have:

(i) When the string is homogeneous, then $\rho_0(x)$ is constant;

(ii) When the string is perfectly elastic, then $\tau_0(t)$ is constant.

4.2 Initial and Boundary Conditions in 1-D String

With the *wave equation*, we want to analyze the energy in 1-D string. In the *wave equation* case, we have the total energy as the sum of total *kinetic energy during vibrations* and the *potential energy* stored in the string.

Respectively, the kinetic energy is:

$$E_{\text{kin}}(t) = \frac{1}{2}\int_0^L \rho_0(x)u_t^2(x,t)dx.$$

Then, by *Hooke's law*, the potential energy stored is:

$$E_{\text{pot}}(t) = \frac{1}{2}\int_0^L \tau_0(t)u_x^2(x,t)dx.$$

Proposition 4.2.1. Total Energy in 1-D String.

Given the *wave equation*, the sum of the energy in the 1-D string system is:

$$E(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t) = \frac{1}{2}\int_0^L \left[\rho_0(x)u_t^2(x,t) + \tau_0(t)u_x^2(x,t)\right] dx.$$

With a formulation for the energy, we then want to learn about when the energy is conserved. Our focus then becomes $E'(t)$, which we have:

$$E'(t) = \int_0^L [\rho_0(x)u_t(x,t)u_{tt}(x,t) + \tau_0(t)u_x(x,t)u_{xt}(x,t)]dx.$$

Utilizing integration by parts for the second part, we have:

$$E'(t) = \int_0^L [\rho_0(x)u_{tt}(x,t) - \tau_0(t)u_{xx}(x,t)]u_t(x,t)dx + \tau_0(t)[u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)].$$

By substituting in the *heat equation*, we have reached that:

$$E'(t) = \int_0^L \rho_0(x)f(x,t)u_t(x,t)dx + \tau_0(t)[u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)].$$

If the energy is conserved in the system, we must have $E'(t) \equiv 0$, which leads us to the conditions of energy-conserved system.

Proposition 4.2.2. Conditions for Energy-conserved System.

The energy is conserved in a 1-D string when the following conditions are both satisfied:

(i) $f(x,t) \equiv 0$, which implies that there are no external force;

(ii) Either $u_x(0,t) = u_x(L,t) = 0$ or $u_t(0,t) = u_t(L,t) = 0$, which implies the orthogonality on the edge or fixed end points.

Hereby, the energy mode and conservation give rise to the construction of initial and boundary conditions.

Definition 4.2.3. Initial Condition for 1-D Wave Equation.

The initial condition for 1-D, bounded rod is:

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x), \qquad \text{for } x \in [0,L].$$

Noe that in our definition, as there are second derivative with respective, thus alike initial conditions in ODEs, we also need the data on the derivative with respect to t to account for the additional information.

Definition 4.2.4. Boundary Condition for 1-D Wave Equation.

The boundary condition for 1-D, bounded rod is encapsulated as follows:

(i) Dirichlet Boundary Condition:

$$u(0,t) = a(t), u(L,t) = b(t), \text{ with } t > 0;$$

(ii) Neumann Boundary Condition:

$$-\tau_0(t)\partial_x u(0,t) = a(t), \tau_0(t)\partial_x u(L,t) = b(t), \text{ with } t > 0;$$

(iii) Robin Boundary Condition:

$$\tau_0(t)u_x(0,t) = ku(x,t), \tau_0(t)u_x(L,t) = -ku(L,t), \text{ with } t > 0.$$

The homogeneous data for Dirichlet and Neumann Boundary Condition is when $a(t) = b(t) \equiv 0$.

Note that there is no boundary conditions assigned for the Global Cauchy problem. Although the string of infinite length seems unrealistic, its initial condition is:

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x), \qquad \text{for } x \in \mathbb{R}.$$

4.3 Solutions to 1-D Wave Equation

In solving the 1-D *wave equation*, we still utilize the method of separation to split the variables to desired states.

Example 4.3.1. Solution to 1-D Wave Equation with Dirichlet's Boundary Condition.

Let the 1-D *wave equation* with Dirichlet's boundary condition be:

$$\begin{cases} \text{PDE:} & u_{tt} - c^2 u_{xx} = 0, \text{ where } x \in (0,L); \\ \text{I.C.:} & u(x,0) = g(x), \qquad u_t(x,0) = h(x), \text{ where } x \in [0,L], \\ \text{B.C.:} & u(0,t) = u(L,t) = 0, \text{ where } t \geq 0. \end{cases}$$

In solving the equation, we assume that:

$$u(x,t) = w(t) \cdot v(x),$$

and insertion back to the *wave equation* exerts that:

$$0 = u_{tt} - c^2 u_{xx} = w''(t)v(x) = c^2 w(t)v''(x),$$

which allows us to set:

$$\frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)} = \lambda \Longrightarrow \begin{cases} w''(t) - \lambda c^2 w(t) = 0, \\ v''(x) - \lambda v(x) = 0, \qquad v(0) = v(L) = 0. \end{cases}$$

Then, we have reduced $v(x)$ into a eigenvalue problem. There are three possibilities for the integral, which are:

(i) If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution;

(ii) If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;

(iii) Eventually, if $\lambda = -\mu^2 < 0$, then we have that:

$$v(x) = A\sin(\mu x) + B\cos(\mu x),$$

and the initial conditions gives us that:

$$\begin{cases} v(0) = B = 0, \\ v(L) = A\sin(\mu L) + B\cos(\mu L) = 0. \end{cases} \Longrightarrow A \text{ is arbitrary, and } B = 0, \text{ with } \mu L = m\pi \text{ for } m = 1, 2, \cdots.$$

Overall, the only non-trivial solution would be:

$$v_m(x) = A\sin(\mu_m x), \qquad \mu_m = \frac{m\pi}{L}.$$

Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the general solution as:

$$w_m(t) = C\cos(\mu_m ct) + D\sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions, which is:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = g(x) = \sum_{m=1}^{\infty} \hat{g_m} \sin(\mu_m x),$$
$$u_t(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = h(x) = \sum_{m=1}^{\infty} \hat{h_m} \sin(\mu_m x),$$

for $x \in [0, L]$. Thus, but the Fourier sine series in the interval, we have:

$$a_m = \hat{g_m} = \frac{2}{L}\int_0^L f(x)\sin(\mu_m x)dx \text{ and } b_m = \frac{\hat{h_m}}{\mu_m c} = \frac{2}{\mu_m cL}\int_0^L h(x)\sin(\mu_m x)dx,$$

which satisfied the system of heat equations.

Remark 4.3.2. Harmonics from the Wave Equation.

While constructing prior to the principle of superposition, we have reached that:

$$u_m(x,t) = [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m t),$$

and this is known as the m-th normal mode of vibration or m-th harmonic, representing a standing wave with frequency of $mc/2L$.

Since we can write the formula of the harmonic as:

$$u_m(x,t) = a_m \sin(\mu_m t)\cos(\mu_m ct) + b_m \sin(\mu_m t)\sin(\mu_m ct),$$

we may use the trigonometric identities, which gives us that:

$$\sin\left(\frac{n\pi}{L}t\right)\sin\left(\frac{m\pi}{L}ct\right) = \frac{1}{2}\cos\left[\frac{m\pi}{L}(x-ct)\right] - \frac{1}{2}\cos\left[\frac{m\pi}{L}(x+ct)\right],$$

thus decompose the harmonics as:

$$u_m(x,t) = R_m(x-ct) + S_m(x+ct),$$
$$u(x,t) = R(x-ct) + S(x+ct),$$

known as *d'Alembert* equation separating the wave traveling to the right and left, respectively, and the velocity is c and $-c$, respectively.

4.4 Uniqueness in 1-D Wave Equation

The proceeding solution is a solution to the 1-D wave equation with Dirichlet's Boundary Condition, and again, we want to know if the solution is unique. Therefore, we want to show the uniqueness of the solution for the wave equation.

Theorem 4.4.1. Uniqueness Solution for 1-D Wave Equation.

The 1-D Wave Equation has unique solution.

In demonstrating the uniqueness, we use the conservation of energy to show. Let u, v be solutions of the wave equation, and let $w = u - v$, and w must be a solution of the same problem with zero initial and boundary conditions, *i.e.*:

$$f(x,t)=0,\ w(x,0)=w_x(x,0)=0,\text{ and } w_t(0,t)=w_t(L,t)=0.$$

From the energy formula, we have that:

$$E'(t)=0 \implies E(t)=E(0)=0 \text{ for every } t\geq 0.$$

By the fact that $E_{\text{kin}}, E_{\text{pot}} \geq 0$, this implies that both kinetic and potential energy must be zero, thus w must be constant and by $w(x,0)=0$, we have $w(x,t)=0$ for every $t\geq 0$. Hence, the solution must be unique.

The above results can be concluded into the stability estimate.

Theorem 4.4.2. Stability Estimate of Wave Equation.

Let $g\in C^4([0,L])$ and $h\in C^3([0,L])$ satisfying conditions:

$$\begin{cases} g(0)=g(L)=g''(0)=g''(L)=0, \\ h(0)=h(L)=0, \end{cases}$$

then the wave equation has a unique solution $u\in C^2([0,L]\times\mathbb{R}^T)$.

Furthermore, u satisfies the stability estimate:

$$\|u(x,t)\|_{0,\infty}^2 \leq 2\max\left\{1, \left(\frac{L}{\pi c}\right)^2\right\}[\|g\|_0^2+\|h\|_0^2],$$

where:

$$\|\varphi(x)\|_0^2=\int_0^L|\varphi(x)|^2dx \text{ and } \|\varphi(x,t)\|_{0,\infty}=\sup_{t>0}\int_0^L|u(x,t)|^2dx.$$

At the same time, we are also introduced to the damping model for the wave equation.

Example 4.4.3. Damping System of 1-D Wave.

Consider a slightly damped vibrating string that satisfied:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}-\alpha\dfrac{\partial u}{\partial t} & x\in(0,L), t>0; \\ u(x,0)=f(x), \qquad \partial_t u(x,0)=g(x) & x\in[0,L]; \\ u(0,t)=u(L,t)=0 & t\geq 0; \end{cases}$$

with $\alpha<\frac{2\pi}{L}$.

Here we attempt to solve using the method of separation, we let:

$$u(x,t)=v(x)w(t),$$

then we have that:

$$v(x)w''(t)=v''(x)w(t)-\alpha v(x)w'(t),$$

which we may assume to separate that:

$$\frac{w''(t)+\alpha w'(t)}{w(t)}=\frac{v''(x)}{v(x)}=\lambda.$$

By $v''(t)-\lambda v(t)=0$, we know that when $\lambda\geq 0$, we will ave the trivial solution. Then, let $\lambda<0$, we have:

$$\lambda=-\mu^2 \Longrightarrow v(t)=A\sin(\mu x)+B\cos(\mu x),$$

then the eigenvalues, by the boundary conditions, we have $B = 0$, are:

$$\mu_m = \frac{m\pi}{L} \Longrightarrow \lambda_m = -\frac{m^2\pi^2}{L^2}.$$

Then, consider the characteristic equation for $w(t)$, which is $r^2 - \alpha r - \lambda_m = 0$, we have the determinant as:

$$\Delta_m = \alpha^2 - 4\frac{m^2\pi^2}{c^2}.$$

Note that if $\Delta_m \geq 0$, we will no longer have the oscillating behavior, thus we must have $\alpha < \frac{2\pi}{L}$.
Given this range, we have:

$$r = \frac{-\alpha \pm iw_m}{2}, \text{ where } w_m = \sqrt{-\Delta m},$$

thus, the solution would be:

$$w_m = Ce^{-\alpha t/2}\sin(w_m t) + De^{-\alpha t/2}\cos(w_m t)$$

Therefore, our solution is now:

$$u = e^{-\alpha t/2}\sum_{m=1}^{\infty}(a_m \cos w_m t + b_m \sin w_m t)\sin\frac{m\pi x}{L}.$$

Therefore, with the given initial conditions, we have:

$$a_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}dx,$$

and by taking the derivatives, we have:

$$u_t(x,0) = \sum_{m=1}^{\infty}\left(-\frac{\alpha}{2}a_m + b_m w_m\right)\sin\frac{m\pi x}{L},$$

which gives that:

$$b_m = \frac{\alpha}{2w_m}\alpha_n + \frac{L}{2w_m}\int_0^{:} g(x)\sin\frac{n\pi x}{L}dx.$$

4.5 Wave Equation in 2-D, $[0,L] \times [0,H]$

With the previous foundations to the 1-D wave, our initial inquiry starts with the rectangular plane $[0,L] \times [0,H]$.

Theorem 4.5.1. Wave Equation in 2-D Plane.

Let $u(x,y,t)$ be the displacement formula at point (x,y) at time t, where $0 \leq x \leq L$, $0 \leq y \leq H$, and $t \geq 0$, their relationship is:

$$u_{tt}(x,y,t) = c^2\nabla^2 u(x,y,t) = c^2\big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\big).$$

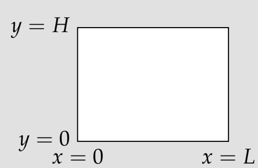

Figure 4.2. Rectangular membrane for 2-D Plane.

Definition 4.5.2. Initial and Boundary Conditions for 2-D Plane.

For the 2-D case, the initial position and velocity would be given as:

$$u(x,y,0)=\alpha(x,y), \qquad \partial_t u(x,y,0)=\beta(x,y).$$

Example 4.5.3. Solution to 2-D Wave Equation with Dirchlet's Boundary Condition..

Let the 2-D *wave equation* with Dirichlet's boundary condition be:

$$\begin{cases} \text{PDE:} & u_{tt}-c^2\nabla^2 u=0, \text{ where } x\in(0,L), y\in(0,H); \\ \text{I.C.:} & u(x,y,0)=\alpha(x,y), \qquad u_t(x,y,0)=\beta(x,y), \text{ where } x\in[0,L], y\in[0,H], \\ \text{B.C.:} & u(0,y,t)=u(L,y,t)=u(x,0,t)=u(x,H,t)=0, \text{ where } t\geq 0. \end{cases}$$

In solving the equation, we assume that:

$$u(x,y,t)=h(t)\phi(x,y)=h(t)f(x)g(y).$$

By plugging the first equality to the wave PDE, we have:

$$\begin{cases} \frac{d^2h}{dt^2}=-\lambda c^2 h; \\ \Delta\phi=-\lambda\phi, \qquad \phi(0,y)=\phi(L,y)=\phi(x,0)=\phi(x,H)=0 \end{cases}$$

Akin to the 1-D case, we have $\lambda>0$, else we would just have the trivial solution. Then, we have $h(t)$ as the linear combination of $\sin(\sqrt{\lambda}t)$ and $\cos(\sqrt{\lambda}t)$.
Then, the second separation gives that:

$$g(y)\frac{d^2f(x)}{dx^2}+f(x)\frac{d^2g(y)}{dy^2}=-\lambda f(x)g(x),$$

so we can let:

$$\frac{f''}{f}=-\lambda-\frac{g''}{g}=-\mu \Longrightarrow \begin{cases} f''+\mu f=0, & f(0)=f(L)=0; \\ g''+(\lambda-\mu)g=0, & g(0)=g(H)=0. \end{cases}$$

First, for variable x and $f(x)$, we have $\mu_n=\left(\frac{n\pi}{L}\right)^2$, $n=1,2,\cdots$, with $f_n(x)=\sin\left(\frac{n\pi x}{L}\right)$.
Likewise, for variable y and $g(y)$, we have $\lambda-\mu_n=\left(\frac{m\pi}{H}\right)^2$, $m=1,2,\cdots$, with $g_m(y)=\sin\left(\frac{m\pi x}{H}\right)$.
Therefore, we can define:

$$\lambda_{nm}=\left(\frac{m\pi}{H}\right)^2+\left(\frac{n\pi}{L}\right)^2,$$

and thus for every pair (x,y):

$$\phi_{mn}(x,y)=f_n(x)g_m(y)=\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{H}\right).$$

Since our $h(x)$ must be the linear combination of the above formula, we shall include all the families, as a nested *Fourier series*, written as:

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\cos(c\sqrt{\lambda_{nm}}t)+B_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin(c\sqrt{\lambda_{nm}}t)\right].$$

The coefficients of the two families can be determined by first having $u(x,y,0)=\alpha(x,y)$, which is:

$$\alpha(x,y)=\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}A_{nm}\sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi y}{H}\right).$$

By decomposing for the Fourier sine coefficients, we have:

$$A_{nm}=\frac{2}{L}\int_0^L\left[\frac{2}{H}\int_0^H\alpha(x,y)\sin\left(\frac{m\pi y}{H}\right)dy\right]\sin\left(\frac{n\pi x}{L}\right)dx.$$

Likewise, for the second having $u_t(x,y,t) = \beta(x,y)$, we have:

$$\beta(x,y) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi y}{H}\right).$$

By also decomposing for the Fourier sine coefficients, we have:

$$c\sqrt{\lambda_{nm}}B_{nm} = \frac{2}{L}\int_0^L\left[\frac{2}{H}\int_0^H \beta(x,y)\sin\left(\frac{m\pi y}{H}\right)dy\right]\sin\left(\frac{n\pi x}{L}\right)dx.$$

Therefore, we have solved our initial value problem with double Fourier series.

Remark 4.5.4. Simplification of Double Integration.

Since we are in the $L^1([0,L]\times[0,H])$ region, we can definitely use *Fubini's theorem* to simplify to a single square region, however, the example demonstrates the method of solving an initial value problem about wave equation.

4.6 d'Alembert Formula and Fundamental Solution to 1D Wave Equation

The d'Alembert formula aims for a solution to the global Cauchy problem for 1-D, namely the following problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}. \end{cases}$$

In finding the solution, we first factorize the wave equation in terms of the operators:

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0,$$

which allows us to define $v(x,t) := u_t + cu_x$, driving the PDE into:

$$(\partial_t - c\partial_x)v = 0 \implies v_t - cv_x = 0,$$

which is the linear transport equation, with the corresponding solution as:

$$v(x,t) = \Psi(x+ct),$$

which leads to:

$$v_t - cv_x = c\Psi'(x+ct) - c\Psi'(x+ct) = 0,$$

where Ψ is any differentiable function. Then, we can plug back into our equation with u, that is:

$$u_t + cu_x = \Psi(x+ct),$$

which, by integration, yields that:

$$u(x,t) = \int_0^t \Psi(x - c(t-s) + cs)ds + \varphi(x-ct),$$

where φ is another arbitrary differentiable function. In particular, the integration is a solution to the problem, which can be verified by the following theorem.

Theorem 4.6.1. Leibniz Integral Rule.

Let the integral be defined as:

$$\int_{a(x)}^{b(x)} f(x,t)dt,$$

where $a(x)$ and $b(x)$ are bounded functions dependent on x, its derivative can be expressed as:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x, b(x)\big)\cdot\frac{db(x)}{dx} - f\big(x,a(x)\big)\cdot\frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt.$$

Note that when we have $a(x)$ and $b(x)$ as constants, we can immediately reduce to:

Corollary 4.6.2. Leibniz Integral Rule with Constant Boundary.

We defined the derivate of the integral with constant bounds is:

$$\frac{d}{dx}\int_a^b f(x,t)dt = \int_a^b \frac{\partial}{\partial x} f(x,t)dt.$$

Returning back to our example, we can shift by $y = x - ct + 2cs$, so we find that:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(y)dy + \varphi(x-ct).$$

Then, we want to determine Ψ and φ with the initial conditions that:

$$\begin{cases} u(x,0) = \varphi(x) = g(x), \\ u_t(x,0) = \Psi(x) - c\varphi'(x) = h(x). \end{cases} \implies \begin{cases} \varphi(x) = g(x), \\ \Psi(x) = h(x) + cg'(x). \end{cases}$$

By inserting back the conditions, we have the *d'Alembert formula* as:

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} \left[h(y) + cg'(y)\right]dy + g(x-ct) \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy + \frac{1}{2c}\left[g(x+ct) - g(x-ct)\right] + g(x-ct) \\ &= \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy. \end{aligned}$$

Noting that we have obtained a solution using the *d'Alembert formula*, and, as conventionally, we are now concerning the uniqueness.

Proposition 4.6.3. Uniqueness for d'Alembert Formula.

The solution to the global Cauchy problem for 1-D, namely:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}; \end{cases}$$

is unique.

The deduction to the uniqueness is from the d'Alembert formula. Let u_1 and u_2 be the solutions with respect to g_1, h_1, and g_2, h_2. Suppose that g_1, h_1, g_2, and h_2 are bounded, then with the *d'Alembert formula*, we have $u_1 - u_2$ satisfies that:

$$|u_1(x,t) - u_2(x,t)| \le \|g_1 - g_2\|_\infty + T\|h_1 - h_2\|_\infty,$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$ where:

$$\|g_1 - g_2\|_\infty = \sup_{x\in\mathbb{R}} |g_1(x) - g_2(x)| \;\text{ and }\; \|h_1 - h_2\|_\infty = \sup_{x\in\mathbb{R}} |h_1(x) - h_2(x)|.$$

Therefore, if $g_1 = g_2$ and $h_1 = h_2$, then we have $|u_1 - u_2|$ bounded above by 0, hence forcing it to be zero. Therefore, we guarantee uniqueness.

Then, we can attempt to decompose the wave into two parts

Definition 4.6.4. Progressive Wave.

First, we define that:

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(y)dy, \qquad G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(y)dy,$$

hence allowing us to write the solution as:

$$u(x,t) = F(x+ct) + G(x-ct).$$

Thus, we are able to give $u(x,t)$ as a superposition of two progressive waves moving at a constant speed of x in both positive and negative x-direction.

At the same time, we consider the progressive waves as the lines of *characteristics*. In particular, we can construct the parallelogram of the progressive waves.

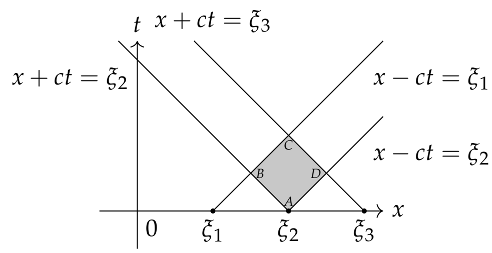

Figure 4.3. Parallelogram of Waves.

Proposition 4.6.5. Waves can be Traced Back.

For each parallelogram of the progressive waves, the following relationship holds:

$$u(A) + u(C) = u(B) + u(D).$$

Namely, we know the information of for each vertices when we know 3 out of the 4 vertices.

Consider that each vertices are on two progressing waves, so we have:

$$F(A) = F(B), \qquad F(C) = F(D), \qquad G(A) = G(D), \qquad G(B) = G(C),$$

hence the above relationship holds for all $u(x) = F(x) + G(x)$.

Meanwhile, waves can be traced back and the waves thus forming "the light cone," leading to the following definitions.

Definition 4.6.6. Domain of Dependence and Range of Influence.

The *domain of dependence* of (x,t) is the interval:

$$[x - ct, x + ct],$$

and the *range of influence* of z entails the region for (x,t) such that:

$$z - ct \le x \le z + ct.$$

This can be described in a image as:

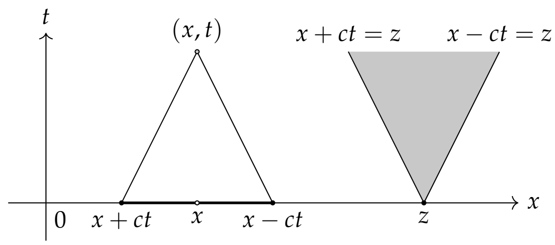

Figure 4.4. Domain of Dependence (bold line) for (x,t) and Range of Influence (shaded region) for z.

With the current *d'Alembert formula*, we can differentiate the last term with respect to time, which gives us that:

$$\frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy\right] = \frac{1}{2c}\left[ch(x+ct) + ch(x-ct)\right] = \frac{1}{2}\big[h(x+ct) + h(X-ct)\big].$$

Then, we can write the formula as:

$$u(x,t) = \frac{\partial}{\partial t}\left[w_g(x,t)\right] + w_h(x,t),$$

where w_f denotes the solution to the problem:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & x \in \mathbb{R}, t > 0; \\ w(x,0) = 0,\ w_t(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Getting sufficiently introduced to the *d'Alembert formula*, we want to construct the fundamental solution, which is built on the *Dirac delta function*.

Definition 4.6.7. Dirac Delta Function.

The Dirac delta function $\delta(x)$ for 1D is zero everywhere except at the origin, and infinite at the origin, *i.e.*:

$$\delta(x) := \begin{cases} +\infty, & x = 0; \\ 0, & x = 0, \end{cases}$$

while satisfying that:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

In particular, we consider the *Dirac delta function* at a point ξ as $\delta(x-\xi)$.

Remark 4.6.8. Dirac Delta Function in $\mathbb{R}^d$.

In d-dimensional Euclidean Space, for any $\mathbf{x} = (x_1, x_2, \cdots, x_d)$, the Dirac delta function is defined as:

$$\delta_d(\mathbf{x}) = \delta(x_1)\cdot\delta(x_2)\cdots\delta(x_d).$$

In particular, the definition can be consider as:

$$\delta_d(\mathbf{x}) := \begin{cases} +\infty, & \mathbf{x} = \mathbf{0}; \\ 0, & \mathbf{x} \neq \mathbf{0}. \end{cases}$$

Our inquiry on the *Fundamental solution* is on the special case of the global Cauchy problem, namely:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & x \in \mathbb{R}, t > 0; \\ w(x,0) = 0,\ w_t(x,0) = \delta(x-\xi), & x \in \mathbb{R}. \end{cases}$$

In particular, we have ξ as the localized unit impulse.

Definition 4.6.9. 1D Wave Kernel.

Particularly, the solution to the above system, or *1D wave kernel*, can be formulated as:

$$\begin{aligned} K(x,\xi,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(y-\xi)dy \\ &= \frac{1}{2c}\big[H(x-\xi+ct) - H(x-\xi-ct)\big] = \frac{1}{2c}\chi_{[-ct,ct]}(x-\xi), \end{aligned}$$

which can be described as:

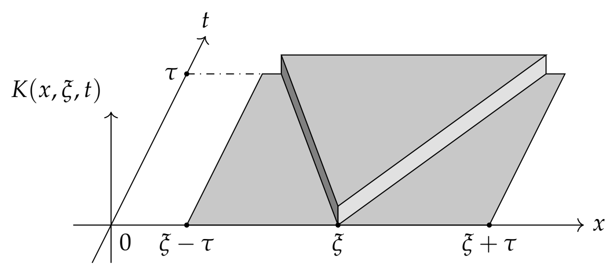

Figure 4.5. The fundamental solution $K(x,\xi,t)$ for $t \in (0,\tau]$.

Conversely, we can construct the solution to global Cauchy problem with the initial conditions as:

$$w(x,0) = 0, \text{ and } w_t(x,0) = h(x), \text{ for } x \in \mathbb{R},$$

where we have that:

$$h(x) = \int_{-\infty}^{\infty} \delta(x-\xi)h(\xi)d\xi.$$

By inputing the values, we have that:

$$\begin{aligned} w_h(x,t) &= \int_{-\infty}^{\infty} K(x,\xi,t)h(\xi)d\xi \\ &= \frac{1}{2c}\int_{-\infty}^{\infty} \chi_{[-ct,ct]}(x-\xi)h(\xi)d\xi = \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)d\xi, \end{aligned}$$

which aligns with the *d'Alembert formula*.

Example 4.6.10. Non-homogeneous Case.

Given a non-homogeneous problem as:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x \in \mathbb{R}, t > 0; \\ u(x,0) = 0,\ u_t(x,0) = 0, & x \in \mathbb{R}; \end{cases}$$

where f and f_x are continuous on $\mathbb{R} \times [0m + \infty)$. Now, we can use the *Duhamel's method*. When having $s \geq 0$, we let $w(x,t;s)$ be the solution of the following system:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & x \in \mathbb{R}, t > 0; \\ w(x,s;s) = 0,\ w_t(x,s;s) = f(x,s), & x \in \mathbb{R}. \end{cases}$$

By the fundamental solution to the global Cauchy problem, we have that:

$$w(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy.$$

Hence, we have the solution to u as:

$$u(x,t) = \int_0^t w(x,t;s)ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds = \frac{1}{2c}\int_{S_{x,t}} f(y,s)dyds,$$

where $S_{x,t}$ is the triangular section, or the domain of dependence with its interior.
The initial conditions for $u(x,t)$ can be verified immediately by plugging in the value of 0 for u and u_t.

4.7 Fundamental Solution in $\mathbb{R}^3$

In the space of the odd dimensions (except 1D), the waves would obey a certain property for the "sharp" initial state, and the phenomenon results in the reappearance of the state, *i.e.*, the sharp leading or trailing edges, at a later moment.

Proposition 4.7.1. Huygen's Principle.

In a space of odd dimension (except 1D) involves the reappearance of a sharply-localized initial state at a later moment of time at another point as a phenomenon just as sharply localized.
If the space has even dimensions, Hyugen's principle is absent.

In particular, we can investigate the dimension 1 case first. The following two example of dimension 1 have shar-localized initial state, and they would demonstrate that (strong) Huygen's principle does not necessarily hold in dimension 1.

Example 4.7.2. 1D Wave with $g(x) = \chi_{[-1,1]}(x)$ and $h(x) = 0$.

Here, we can formulate its solution on the right side (say $x \geq 0$) as:

$$u(x_0,t) = \begin{cases} 0, & \text{if } 0 \leq t < \dfrac{x_0-1}{c}, \\ \dfrac{1}{2}, & \text{if } \dfrac{x_0-1}{c} \leq t \leq \dfrac{x_0+1}{c}, \\ 0, & \text{if } t > \dfrac{x_0+1}{c}. \end{cases}$$

Visually, the propagation of the wave looks like:

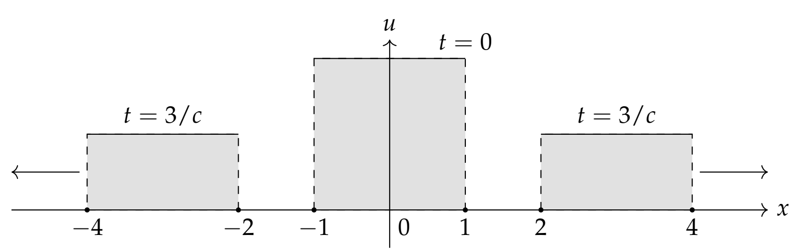

Figure 4.6. Propagation of the Wave in 1D following Huygen's Principle.

Here, we could notice that for each x_0, we have the wave hitting and leaving sharply, and it travels with the velocity c.

The above example satisfies the Huygen's principle, but the following is a counter example disproving Huygen's principle in 1D.

Example 4.7.3. 1D Wave with $g(x) = 0$ and $h(x) = \chi_{[-1,1]}(x)$.

Here, we would have the solution on the right side differently, as:

$$u(x_0, t) = \begin{cases} 0, & \text{if } 0 \le t < \dfrac{x_0 - 1}{c}, \\ \dfrac{1}{2c}\displaystyle\int_{x_0 - ct}^{1} d\xi = \dfrac{1 - (x_0 - ct)}{2c}, & \text{if } \dfrac{x_0 - 1}{c} \le t \le \dfrac{x_0 + 1}{c}, \\ \dfrac{1}{c}, & \text{if } t > \dfrac{x_0 + 1}{c}. \end{cases}$$

Visually, the propagation of the wave looks like:

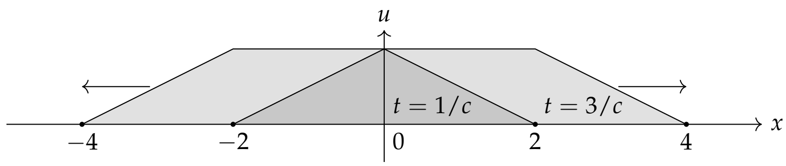

Figure 4.7. Propagation of the Wave in 1D not following Huygen's Principle.

We notice that the leading edge is sharp (entering is gradual, but still sharp), but there is no trailing edge in this case.

Now, we will take a break from the Huygen's principle, but get our focus on the solution to 3D homogeneous wave equation, as follows:

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \mathbf{x} \in \mathbb{R}^3,\ t > 0; \\ u(\mathbf{x}, 0) = g(\mathbf{x}),\ u_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Akin to the 1D case, the uniqueness of the solution guarantees that we have a single $u \in C^2(\mathbb{R}^3 \times [0, +\infty])$, our goal is to show that a solution u exists in terms of g and h.

Our first step is to separate g and h apart. Again, we denote w_h as the solution to the following problem:

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, & x \in \mathbb{R}^3,\ t > 0; \\ w(\mathbf{x}, 0) = 0,\ w_t(\mathbf{x}, 0) = h(\mathbf{x}), & x \in \mathbb{R}^3. \end{cases}$$

Theorem 4.7.4. Solution to Non-trivial Initial Position.

If $w_g \in C^3(\mathbb{R}^3 \times [0, +\infty))$, then $v = \partial_t w_g$ solves the following problem:

$$\begin{cases} v_{tt} - c^{\Delta} v = 0, & \mathbf{x} \in \mathbb{R}^3,\ t > 0; \\ v(\mathbf{x}, 0) = g(\mathbf{x}),\ v_t(\mathbf{x}, 0) = 0, & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

The verification is immediate by checking the derivatives of v, that is:

$$v_{tt} - c^2 \Delta v = (\partial_{tt} - c^2 \Delta)\partial_t w_g = \partial_t(\partial_{tt} w_g - c^2 \Delta w_g) = 0.$$

Then, as long as we check the initial conditions that:

$$v(\mathbf{x}, 0) = \partial_t w_g(\mathbf{x}, 0) = g(\mathbf{x}), \text{ and } v_t(\mathbf{x}, 0) = \partial_{tt} w_g(\mathbf{x}, 0) = c^2 \Delta w_g(\mathbf{x}, 0) = 0.$$

Hence, we can verify that the construction of v is a solution to the non-trivial initial position case.

With the above result, when we return to the solution to the 3D problem, the unique solution is:

$$u(\mathbf{x}, t) = \partial_t w_g(\mathbf{x}, t) + w_h(\mathbf{x}, t).$$

Again, we want to focus on the special case when $h(\mathbf{x}) = \delta_3(\mathbf{x})$, *i.e.*, the Fundamental solution. This case can be considered as the case of sound waves, when there is a sudden change of the air density at the origin. The case is:

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, & \mathbf{x} \in \mathbb{R}^3,\ t > 0; \\ w(\mathbf{x}, t) = 0,\ w_t(\mathbf{x}, 0) = \delta_3(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Definition 4.7.5. 3D Wave Kernel.

Here, by using the radial symmetry, we conclude that for $t > 0$, we can obtain the solution to the above system, or *wave kernel for 3D*, as follows:

$$K(\mathbf{x}, t) = \frac{\delta(|\mathbf{x}| - ct)}{4\pi c|\mathbf{x}|}.$$

One might be immediately questioning the purpose of deriving the *wave kernel*, or the fundamental solution, but they comes out handy in finding the general solution.

Proposition 4.7.6. Solution to General 3D Wave Global Cauchy Problem.

For arbitrary $h(\mathbf{x})$, we have that:

$$h(\mathbf{x}) = \int_{\mathbb{R}^3} \delta_3(\mathbf{x} - \mathbf{y}) h(\mathbf{y}) d\mathbf{y},$$

hence leading to the solution that:

$$w_h(\mathbf{x}, t) = \int_{\mathbb{R}^3} K(\mathbf{x} - \mathbf{y}, t) h(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^3} \frac{\delta(|\mathbf{x} - \mathbf{y}| - ct)}{4\pi c|\mathbf{x} - \mathbf{y}|} h(\mathbf{y}) d\mathbf{y}.$$

Here, w_h is the $\mathbf{x}$-convolution of h with the fundamental solution, namely:

$$w_h(\mathbf{x}, t) = \int_0^\infty \frac{\delta(r - ct)}{4\pi cr} dr \int_{\partial B_r(\mathbf{x})} h(\sigma) d\sigma = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} h(\sigma) d\sigma.$$

With all of these foundations, we are returning to our main question, which is finding the solution to the general system for 3D waves for global Cauchy problem.

Theorem 4.7.7. Kirchhoff's Formula.

Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$, and let the system of equations being:

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \mathbf{x} \in \mathbb{R}^3,\ t > 0; \\ u(\mathbf{x}, 0) = g(\mathbf{x}),\ u_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

We have the unique solution $u \in C^2(\mathbb{R}^3 \times [0, +\infty))$ to the above system as:

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} g(\sigma) d\sigma \right] + \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} h(\sigma) d\sigma.$$

In particular, we can write the Kirchhoff's formula through a transformation.

Remark 4.7.8. Alternative Kirchhoff's Formula.

One can write the Kirchhoff formula in the following form:

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\mathbf{x})}[g(\boldsymbol{\sigma}) + \nabla g(\boldsymbol{\sigma})\cdot(\boldsymbol{\sigma}-\mathbf{x}) + th(\boldsymbol{\sigma})]d\sigma.$$

Here, we want to transform the ball into unit ball with $\boldsymbol{\sigma} = \mathbf{x} + ct\boldsymbol{\omega}$ so that $\boldsymbol{\omega} \in B_1(\mathbf{0})$, as follows:

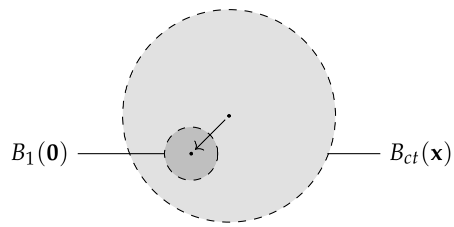

Figure 4.8. Transforming any Ball to the Unit Ball.

Correspondingly, the surface area is varied by the ratio of surface area, which is correlated to the square of radius (which is $S \propto r^{d-1}$):

$$d\sigma = c^2t^2 d\omega.$$

Then, we want to rewrite w_g and w_h as:

$$w_g(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\mathbf{x})} g(\boldsymbol{\sigma})d\sigma = \frac{1}{4\pi c^2 t}\int_{\partial B_1(\mathbf{0})} g(\mathbf{x}+ct\boldsymbol{\omega})c^2t^2 d\omega = \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} g(\mathbf{x}+ct\boldsymbol{\omega})d\omega;$$
$$w_h(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\mathbf{x})} h(\boldsymbol{\sigma})d\sigma = \frac{1}{4\pi c^2 t}\int_{\partial B_1(\mathbf{0})} h(\mathbf{x}+ct\boldsymbol{\omega})c^2t^2 d\omega = \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} h(\mathbf{x}+ct\boldsymbol{\omega})d\omega.$$

Specifically, we want to take the partial derivatives for $w_g(\mathbf{x},t)$, which is:

$$\frac{\partial}{\partial t}w_g(\mathbf{x},t) = \frac{1}{4\pi}\int_{\partial B_1(\mathbf{0})} g(\mathbf{x}+ct\boldsymbol{\omega})d\omega + \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla g(\mathbf{x}+ct\boldsymbol{\omega})\cdot c\boldsymbol{\omega}d\omega.$$

Now, we plug in the information to the original equation, which leads us to the alternative formulation:

$$\begin{aligned}
u(\mathbf{x},t) &= \frac{\partial}{\partial t}w_g(\mathbf{x},t) + w_h(\mathbf{x},t)\\
&= \frac{1}{4\pi}\int_{\partial B_1(\mathbf{0})} g(\mathbf{x}+ct\boldsymbol{\omega})d\omega + \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla g(\mathbf{x}+ct\boldsymbol{\omega})\cdot c\boldsymbol{\omega}d\omega + \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} h(\mathbf{x}+ct\boldsymbol{\omega})d\omega\\
&= \frac{1}{4\pi}\int_{\partial B_1(\mathbf{0})} \left[g(\mathbf{x}+ct\boldsymbol{\omega}) + \nabla g(\mathbf{x}+c\boldsymbol{\omega})\cdot ct\boldsymbol{\omega}d\omega + th(\mathbf{x}+ct\boldsymbol{\omega})\right]d\omega\\
&= \frac{1}{4\pi}\int_{\partial B_{ct}(\mathbf{x})} [g(\boldsymbol{\sigma}) + \nabla g(\boldsymbol{\sigma})\cdot(\boldsymbol{\sigma}-\mathbf{x}) + th(\boldsymbol{\sigma})]\cdot\frac{1}{c^2t^2}d\sigma\\
&= \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\mathbf{x})}[g(\boldsymbol{\sigma}) + \nabla g(\boldsymbol{\sigma})\cdot(\boldsymbol{\sigma}-\mathbf{x}) + th(\boldsymbol{\sigma})]d\sigma.
\end{aligned}$$

Remark 4.7.9. Alignment with the Domain of Dependence.

The above result indicates that $u(\mathbf{x},t)$ depends upon $g(\mathbf{x})$ and $h(\mathbf{x})$ only for $\mathbf{x} \in \partial B_{ct}(\mathbf{x})$, which is the surface of the ball. This coincides with the domain of dependence for $(\mathbf{x},t)$.

Finishing up our argument on the general 3D Cauchy problem for wave equations. We will return to the Huygen's principle. In particular, the space of dimension 3 is a space where the (strong) Huygen's principle holds.

Proposition 4.7.10. Strong Huygen's Principle in 3D.

Suppose that g and h have a compact support D, then $u(\mathbf{x}, t)$ differs from 0 only on some open interval $(t_{\min}, t_{\max})$, in which they are the first and last time t such that:

$$D \cap \partial B_{ct}(\mathbf{x}) \neq \emptyset.$$

I.e., a disturbance that is initially localized in D starts affect the point $\mathbf{x}$ at time $t_{\min}$ and ends its effect after time $t_{\max}$.

Then, if we consider that $w_t(\mathbf{x}, 0) = \delta_3(\mathbf{x} - \mathbf{y})$, where $\mathbf{y}$ is the center of the initial datum. By the spatial invariance, its corresponding fundamental solution is:

$$K(\mathbf{x} - \mathbf{y}, t) = \frac{\delta(|\mathbf{x} - \mathbf{y}| - ct)}{4\pi c|\mathbf{x} - \mathbf{y}|}, \ t > 0,$$

which resembles a outgoing traveling wave supported on:

$$\partial B_{ct}(\mathbf{y}) = \{\mathbf{x} : |\mathbf{x} - \mathbf{y}| = ct\}.$$

In particular, we have the support of $K(\mathbf{x} - \mathbf{y}, t)$ being:

$$\operatorname{supp}(K) := \bigcup_{t>0} \partial B_{ct}(\mathbf{y}) = \{(\mathbf{x}, t) : |\mathbf{x} - \mathbf{y}| \leq ct, t > 0\} =: C^*_{\mathbf{y},0}.$$

Remark 4.7.11. Range of Influence for 3D Waves.

The range of influence for $\mathbf{y}$ is $\partial C^*_{\mathbf{y},0}$, it is only the boundary of the forward of the cone, not interior.

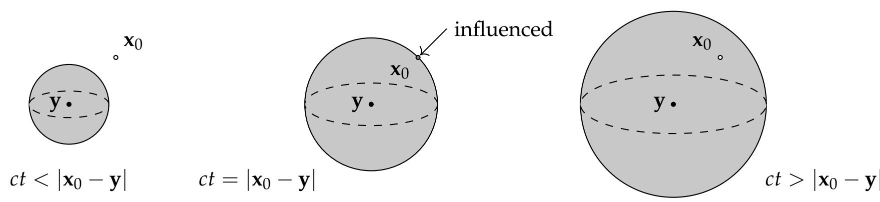

Figure 4.9. Huygen's Principle in 3D, Propagation of Waves.

4.8 Method of Descent

Then, we want to move our attention to the dimension in between, the Cauchy problem for 2D waves.

Remark 4.8.1. 2D Cauchy Problems are Inseparable.

In 2D case, when we convert to polar coordinates, we have that:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} u\right) = c^2 \left(\partial_{rr} u + \frac{1}{r} \partial_r u\right).$$

Since we cannot separate the operators, we cannot transfer the current equation to the 1 dimension case.

Therefore, we cannot apply Kirchhoff formula for 2D waves, however, we can use an alternating approach by embedding the 2D waves into three dimensional system.

Example 4.8.2. Hadamard's Method of Descent.

By *Hadamard's Method of Descent*, with the two-dimension problem:

$$\begin{cases} w_{tt}(\mathbf{x}, t) - c^2 \Delta w(\mathbf{x}, t) = 0, & \mathbf{x} \in \mathbb{R}^2, \ t > 0; \\ w(\mathbf{x}, 0) = 0, \ w_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2. \end{cases}$$

We embed the equations into three dimensions, namely:

$$\begin{cases} w_{tt}(\mathbf{x}, x_3, t) - c^2\Delta w(\mathbf{x}, x_3, t) = 0, & (\mathbf{x}, x_3) \in \mathbb{R}^3,\ t > 0; \\ w(\mathbf{x}, x_3, 0) = 0,\ w_t(\mathbf{x}, x_3, 0) = h(\mathbf{x}), & (\mathbf{x}, x_3) \in \mathbb{R}^3. \end{cases}$$

Here, by using the 3D *Kirchhoff formula*, we have that:

$$U(\mathbf{x}, x_3, t) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x}, x_3)} h d\sigma.$$

Since h is independent from x_3, we pick $x_3 = 0$. Moreover, consider the spherical surface as the upper and lower hemisphere, we have them as:

$$y_3 = x_3 \pm \sqrt{c^2t^2 - r^2}.$$

Note that $r = |\mathbf{y} - \mathbf{x}|$, we can let both hemispheres as:

$$d\sigma = \sqrt{1 + |\nabla y_3|^2} d\mathbf{y} = \sqrt{1 + \frac{r^2}{c^2t^2 - r^2}} d\mathbf{y} = \frac{ct}{\sqrt{c^2t^2 - r^2}} d\mathbf{y}.$$

Hence, we can write that:

$$U(\mathbf{x}, t) = U(\mathbf{x}, x_3, t) = \frac{1}{2\pi c} \int_{B_{ct}(\mathbf{x})} \frac{h(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{x} - \mathbf{y}|^2}} d\mathbf{y}.$$

The above example leads to the general formula for 2D Cauchy problem for waves.

Theorem 4.8.3. Poisson's Formula.

Let $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$, and with the system being:

$$\begin{cases} u_{tt} - c^2\Delta u = 0, & \mathbf{x} \in \mathbb{R}^2,\ t > 0; \\ u(\mathbf{x}, 0) = g(\mathbf{x}),\ u_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2. \end{cases}$$

It has a unique solution $u \in C^2(\mathbb{R}^2 \times [0, +\infty))$, namely:

$$u(\mathbf{x}, t) = \frac{1}{2\pi c} \left[\frac{\partial}{\partial t} \int_{B_{ct}(\mathbf{x})} \frac{g(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{x} - \mathbf{y}|^2}} d\mathbf{y} + \int_{B_{ct}(\mathbf{x})} \frac{h(\mathbf{y})}{\sqrt{c^2t^2 - |\mathbf{x} - \mathbf{y}|^2}} d\mathbf{y} \right].$$

Remark 4.8.4. Fundamental Solution to 2D Waves.

By using the Poisson's formula, the fundamental solution for the two dimension wave equation is:

$$K(\mathbf{x}, t) = \frac{1}{2\pi c} \cdot \frac{1}{\sqrt{c^2t^2 - |\mathbf{x}|^2}} \chi_{B_{ct}(\mathbf{x})}(\mathbf{x}).$$

In fact, we can use the *method of descent* to obtain a lower dimensional system. The following is an example of descending a dimensional 2 system to a dimensional 1 system.

Example 4.8.5. Descending to Dimension 1 System.

Here, we "embed" the one-dimensional problem into a two-dimensional setting, which is:

$$\begin{cases} u_{tt}(x, y, t) - c^2\Delta u(x, y, t) = 0, & (x, y) \in \mathbb{R}^2, t > 0; \\ u(x, y, 0) = g(x),\ u_t(x, y, 0) = h(x), & (x, y) \in \mathbb{R}^2. \end{cases}$$

Note that by Poisson's formula, we know the solution to the two-dimensional problem is:

$$u(x,y,t)=\frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\int_{B_{ct}(x,y)}\frac{g(x',y')d(x',y')}{\sqrt{c^2t^2-|(x,y)-(x',y')|^2}}+\int_{B_{ct}(x,y)}\frac{h(x',y')d(x',y')}{\sqrt{c^2t^2-|(x,y)-(x',y')|^2}}\right].$$

First, we want to make a shift in the equations and integral regions, which does not make a change:

$$u(x,y,t)=\frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\int_{B_{ct}(\mathbf{0})}\frac{g(x+x',y+y')}{\sqrt{c^2t^2-|(x',y')|^2}}d(x',y')+\int_{B_{ct}(\mathbf{0})}\frac{h(x+x',y+y')}{\sqrt{c^2t^2-|(x',y')|^2}}d(x',y')\right].$$

Since we claim that g and h are independent on y, we have u independent of y as well:

$$u(x,y,t)=\frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\int_{B_{ct}(\mathbf{0})}\frac{g(x+x')}{\sqrt{c^2t^2-|(x',y')|^2}}d(x',y')+\int_{B_{ct}(\mathbf{0})}\frac{h(x+x')}{\sqrt{c^2t^2-|(x',y')|^2}}d(x',y')\right].$$

Note that we want to make the limits of integration into x and y, which we need to switch the boundary for $B_{ct}(\mathbf{0})$, which is $-ct\le x\le ct$ and $-\sqrt{c^2t^2-x^2}\le y\le\sqrt{c^2t^2-x^2}$, we we have that:

$$\begin{aligned}u(x,y,t)=\frac{1}{2\pi c}\Bigg[&\frac{\partial}{\partial t}\int_{-ct}^{ct}g(x+x')\int_{-\sqrt{c^2t^2-(x')^2}}^{\sqrt{c^2t^2-(x')^2}}\frac{1}{\sqrt{c^2t^2-|(x',y')|^2}}dy'dx'\\&+\int_{-ct}^{ct}h(x+x')\int_{-\sqrt{c^2t^2-(x')^2}}^{\sqrt{c^2t^2-(x')^2}}\frac{1}{\sqrt{c^2t^2-|(x',y')|^2}}dy'dx'\Bigg].\end{aligned}$$

Here, we analyze the inner integrand, which is:

$$\begin{aligned}\int_{-\sqrt{c^2t^2-(x')^2}}^{\sqrt{c^2t^2-(x')^2}}\frac{1}{\sqrt{c^2t^2-|(x',y')|^2}}dy'&=\int_{-\sqrt{c^2t^2-(x')^2}}^{\sqrt{c^2t^2-(x')^2}}\frac{1}{\sqrt{c^2t^2-(x')^2-(y')^2}}dy'\\&=\arcsin\left(\frac{y'}{\sqrt{c^2t^2-(x')^2}}\right)\Bigg|_{y'=-\sqrt{c^2t^2-(x')^2}}^{y'=\sqrt{c^2t^2-(x')^2}}\\&=\arcsin(1)-\arcsin(-1)=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi.\end{aligned}$$

Hence, our above integration evaluates to:

$$\begin{aligned}u(x,y,t)&=\frac{\pi}{2\pi c}\left[\frac{\partial}{\partial t}\int_{-ct}^{ct}g(x+x')dx'+\int_{-ct}^{ct}h(x+x')dx'\right]=\frac{1}{2c}\left[cg(x+ct)-cg(x-ct)+\int_{-ct}^{ct}h(x+x')dx'\right]\\&=\frac{1}{2}[g(x+ct)-g(x-ct)]+\frac{1}{2c}\int_{-ct}^{ct}h(x+t)dt,\end{aligned}$$

which aligns with the d'Alembert formula.

Coming back to *Huygen's principle*. Clearly, dimension 2 is an even dimension, so we should anticipate it not necessarily holding.

Remark 4.8.6. Domain of Dependence for 2D Cauchy Problem is Full Circle.

In particular, the *domain of dependence* of the point $(\mathbf{x},t)$ is the full circle, namely:

$$B_{ct}(\mathbf{x})=\{\mathbf{y}:|\mathbf{x}-\mathbf{y}|<ct\}.$$

Therefore, it is noticeable that we have a disturbance, initially localized at $\boldsymbol{\xi}$, and starting the influence from $t>|\mathbf{x}-\boldsymbol{\xi}|/c$, and there is not a end point.

In particular, for the 2D waves, the turbulences will persist after the wave has reached a point. Hence, the sharp signals do not exist in dimensional 2. Specifically, the *(strong) Huygen's Principle* does not hold.

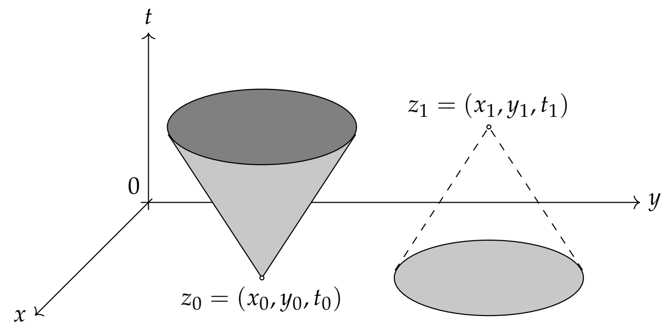

Figure 4.10. The Region of Influence (left) and Domain of Dependence (right).

5 Laplace Equation

5.1 Laplace Equation for Dimension $d \geq 2$

The *Laplace equation* can be interpreted as the *heat equation* at a stationary state, *i.e.*, getting rid of the time variable.

Definition 5.1.1. Laplace/Poisson Equation.

For a *Laplace* (or *Poisson*) *equation*, we have the equation:

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \text{ for } x \in \Omega \subset \mathbb{R}^n,$$

where Ω is a bounded domain, *i.e.*, a non-empty, bounded, connected, and open set in a finite dimensional Euclidean Space.

In particular, one can always contain the bounded domain Ω in some open ball, *i.e.*:

$$\Omega \subset B_r(\mathbf{x}_0),$$

of some $r > 0$ for all $\mathbf{x}_0 \in \mathbb{R}^d$.

Since *Laplace equation* is time independent, it does not contain the *initial condition*. However, it still contains the *boundary condition*, just like other other equations.

Definition 5.1.2. Boundary Conditions for Laplace Equation.

On the boundary $\partial\Omega$, the Laplace equation could be assigned to one of the following boundary conditions:

(i) Dirichlet Boundary Condition:

$$u(\mathbf{x}) = g(\mathbf{x}), \text{ for } \mathbf{x} \in \partial\Omega;$$

(ii) Neumann Boundary Condition:

$$\partial_{\mathbf{n}} u(\mathbf{x}) = h(\mathbf{x}), \text{ for } \mathbf{x} \in \partial\Omega;$$

(iii) Robin Boundary Condition:

$$\partial_{\mathbf{n}} u(\mathbf{x}) + \alpha u(\mathbf{x}) = h, \text{ for } \mathbf{x} \in \partial\Omega \text{ and } \alpha > 0;$$

where $\mathbf{n}$ is the outward normal unit vector to $\partial\Omega$.

In particular, we are introducing two concepts for identifying some classes of Laplace equations.

Definition 5.1.3. Harmonic and Homogeneous Laplace Equations.

A Laplace equation is *harmonic* when its equation satisfies that $f(\mathbf{x}) \equiv 0$ for $\mathbf{x} \in \Omega$, *i.e.*:

$$\Delta u(\mathbf{x}) = 0, \text{ for } \mathbf{x} \in \Omega.$$

A Laplace equation is *homogeneous* when its boundary condition satisfy $g(\mathbf{x}) \equiv 0$ or $h(\mathbf{x}) \equiv 0$, respectively, for $\mathbf{x} \in \partial\Omega$.

Here, we want to investigate the uniqueness of the Laplace equation with its boundary conditions.

Theorem 5.1.4. Uniqueness Solution to the Laplace Equation.

Suppose $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain. There exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying the *Dirichlet* and *Robin boundary condition*.
For the *Neumann boundary condition*, any two solutions differ by a constant.

Likewise to all our prior justifications, we suppose that u and v are two solutions of the same equation and boundary condition, and assign $w := u - v$. Thus w is harmonic and homogeneous.
Recall that we have *Green's First Identity* as:

$$\iint_{\partial W} \varphi(\nabla\psi \cdot \mathbf{n}) dS = \iiint_W [\varphi(\Delta\psi) + (\nabla\varphi) \cdot (\nabla\psi)] dV,$$

and by applying it to the domain Ω and assign $\varphi = \psi = w$, we have that:

$$\int_\Omega |\nabla w|^2 d\mathbf{x} = \int_{\partial\Omega} w \partial_{\mathbf{v}} w d\sigma.$$

In particular, for the *Dirichlet* and *Neumann* conditions, we have:

$$\int_\Omega |\nabla w|^2 d\mathbf{x} = 0,$$

where as for *Robin* condition, we obtain that:

$$\int_\Omega |\nabla w|^2 d\mathbf{x} = -\int_{\partial\Omega} \alpha w^2 d\sigma \leq 0.$$

Since the integral is non-negative, $\partial_{\mathbf{v}} w = \mathbf{0}$, so $w = u - v \equiv c$ for some constant c.
In particular, since *Dirichlet* and *Robin* conditions contains information for u and v, this forces the constant c to be 0, uniquely. While for the *Neumann* condition, since it focuses on the derivatives already, we cannot retrieve the information on the constant.

Remark 5.1.5. Compatibility Condition for Neumann Boundary Condition.

Consider the *Neumann boundary condition*, such that:

$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega; \\ \partial_{\mathbf{n}} u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

By integrating the solution on Ω, we must have that:

$$\int_\Omega f(\mathbf{x}) d\mathbf{x} = \int_\Omega \Delta u(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} \nabla u(\mathbf{x}) \cdot \mathbf{n} d\sigma = \int_{\partial\Omega} h(\mathbf{x}) d\sigma.$$

In particular, a necessary condition for a solvable Laplace equation for Neumann boundary condition is:

$$\int_\Omega f(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} h(\mathbf{x}) d\sigma.$$

Since this is a sufficient condition, so satisfying it does not imply the existence of solution, but violating it guarantees that the equation with boundary condition has no solution.

5.2 Mean Value Property and Maximum Principle

Similar to the *heat equation*, the *Laplace equation* embraces the Maximum principle. Moreover, it also follows the *mean value property*. In prior to the properties, we shall rigorously define the following terms.

Definition 5.2.1. Compact Embedding.

For any topological space X, let $V, W \subset X$, V is compactly embedded in W, *i.e.*, $V \subset\subset W$, when:

(i) $V \subset \overline{V} \subset W^\circ$, where $\overline{V}$ is the closure of V and W° is the interior of W;

(ii) $\overline{V}$ is compact.

In our conditions, any open ball is bounded, so its closure is naturally compact by *Heine-Borel Theorem*. Consider that our bounded domain Ω is non-empty and open, we can always fit open balls compactly contained inside Ω.

Definition 5.2.2. Mean Value Property.

Let $f(\mathbf{x})$ be a real-valued function defined on a open set $\mathcal{O} \subset \mathbb{R}^n$, f satisfied the *mean value property* if for all $B \subset\subset \mathcal{O}$, the value of u at the center is the same as the average of the values of u on the boundary ∂B, *i.e.*, for all $B_r(\mathbf{x}) \subset\subset \mathcal{O}$, f satisfies that:

$$f(\mathbf{x}) = \fint_{\partial B_r(\mathbf{x})} u(\sigma) d\sigma = \fint_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y},$$

where $\fint$ denotes the integral divided by surface area or volume of $B_r(\mathbf{x})$, respectively.

Therefore, we want to establish the connections between the *Mean Value Property* and *Harmonic functions* for the *Laplace equation*.

Theorem 5.2.3. Harmonic $\Longrightarrow$ Mean Value Property.

Let u be harmonic in $\Omega \subset \mathbb{R}^d$, *mean value property* holds for u on Ω.

We start our proof attempting to show, for all $B_R(\mathbf{x}) \subset\subset \Omega$, that:

$$u(\mathbf{x}) = \fint_{\partial B_R(\mathbf{x})} u(\boldsymbol{\sigma}) d\sigma = \frac{1}{S(\partial B_1(\mathbf{0}))R^{d-1}} \int_{\partial B_R(\mathbf{x})} u(\boldsymbol{\sigma}) d\sigma.$$

Here, for $r \le R$, we define:

$$g(r) = \frac{1}{S(\partial B_1(\mathbf{0}))r^{d-1}} \int_{\partial B_r(\mathbf{x})} u(\boldsymbol{\sigma}) d\sigma.$$

Here, we perform the change of variable, which is $\boldsymbol{\sigma} = \mathbf{x} + r\boldsymbol{\sigma}'$, then we have $\boldsymbol{\sigma}' \in \partial B_1(\mathbf{0})$ and $d\sigma = r^{n-1} d\sigma'$, and the function definition becomes:

$$g(r) = \frac{1}{S(\partial B_1(\mathbf{0}))} u(\mathbf{x} + r\boldsymbol{\sigma}') d\sigma'.$$

By taking the derivatives of both sides and applying *Divergence theorem*, we obtain:

$$\begin{aligned} g'(r) &= \frac{1}{S(\partial B_1(\mathbf{0}))} \int_{\partial B_1(\mathbf{0})} \frac{d}{dr} \left[u(\mathbf{x} + r\boldsymbol{\sigma}') \right] d\sigma' = \frac{1}{S(\partial B_1(\mathbf{0}))} \int_{\partial B_1(\mathbf{0})} \nabla u(\mathbf{x} + r\boldsymbol{\sigma}') d\sigma' \\ &= \frac{r}{S(\partial B_1(\mathbf{0}))} \int_{B_1(\mathbf{0})} \Delta u(\mathbf{x} + r\mathbf{y}) d\mathbf{y} = 0. \end{aligned}$$

Therefore, g is a constant, and as we consider $r \to 0$, we have:

$$\lim_{r\to 0} g(r) = \lim_{r\to 0} \frac{1}{S(\partial B_1(\mathbf{0}))} \int_{\partial B_1(\mathbf{0})} u(\mathbf{x} + r\boldsymbol{\sigma}') d\sigma' = \frac{1}{S(\partial B_1(\mathbf{0}))} \int_{\partial B_1(\mathbf{0})} d\sigma' u(\mathbf{x}) = \frac{S(\partial B_1(\mathbf{0}))}{S(\partial B_1(\mathbf{0}))} \cdot u(\mathbf{x}) = u(\mathbf{x}).$$

Hence, we have proven the part where the center is the average of the boundaries:

$$u(\mathbf{x}) = \fint_{\partial B_R(\mathbf{x})} u(\boldsymbol{\sigma})d\sigma.$$

Now, we integrate our established condition for the boundary, in which we have:

$$\begin{aligned}
\left(S(\partial B_1(\mathbf{0}))R^{d-1}\right) \cdot u(\mathbf{x}) &= \int_{\partial B_R(\mathbf{x})} u(\boldsymbol{\sigma})d\sigma \\
\int_0^R \left(S(\partial B_1(\mathbf{0}))s^{d-1}\right) \cdot u(\mathbf{x})ds &= \int_0^R \int_{\partial B_s(\mathbf{x})} u(\boldsymbol{\sigma})d\sigma ds \\
S(\partial B_1(\mathbf{0})) \int_0^R s^{d-1}ds u(\mathbf{x}) = S(\partial B_1(\mathbf{0})) \cdot \frac{R^d}{d} \cdot u(\mathbf{x}) &= \int_{B_R(\mathbf{x})} u(\mathbf{y})d\mathbf{y} \\
u(\mathbf{x}) &= \frac{d}{S(\partial B_1(\mathbf{0})) \cdot R^d} \int_{B_R(\mathbf{x})} u(\mathbf{y})d\mathbf{y} = \fint_{B_R(\mathbf{x})} u(\mathbf{y})d\mathbf{y},
\end{aligned}$$

hence we have shown that u satisfies the *mean value property* on Ω.

The converse of the argument aligns as follows.

Theorem 5.2.4. Bounded & Mean Value Property $\Longrightarrow$ Harmonic.

Let u be bounded on the bounded domain $\Omega \subset \mathbb{R}^d$. If u satisfies the *mean value property*, then $u \in C^\infty(\Omega)$ and it is harmonic on Ω.

Notice that if two functions satisfy the *mean value property* in domain Ω, then their difference satisfies this property as well. Then, suppose that $u \in C(\Omega)$ satisfied the *mean value property* and let a ball $B \subset\subset \Omega$, then we let v be the solution to:

$$\begin{cases} \Delta v(\mathbf{x}) = 0, & \text{for } \mathbf{x} \in B; \\ v(\mathbf{x}) = u(\mathbf{x}), & \text{for } \mathbf{x} \in \partial B. \end{cases}$$

Therefore, we have $v \in C^\infty(B) \cap C(\overline{B})$. Then, we have $w = v - u$ satisfying the mean value property in B, hence attaining its maximum and minimum on ∂B. Since $w = 0$ on ∂B, then we have $u = v$ in B, hence u is harmonic in Ω.

Almost immediately, the previous theorem give rise to the maximum principle for Laplace equation, which corresponds to the maximum property for the heat equation.

Proposition 5.2.5. Maximum Principle for Laplace Equations.

Let $\Omega \subset \mathbb{R}^d$ be a domain and $u \in C(\Omega)$. If u has the mean value property and attains its maximum or minimum at $\mathbf{p} \in \Omega$, then u is constant.
In particular, if Ω is bounded and $u \in C(\overline{\Omega})$ is not constant, then, for every $\mathbf{x} \in \Omega$:

$$u(\mathbf{x}) < \max_{\partial\Omega} u \text{ and } u(\mathbf{x}) > \min_{\partial\Omega} u.$$

Without loss of generality, we suppose that $\mathbf{p} \in \Omega$ is the minimum point for u, then:

$$m = u(\mathbf{p}) \leq u(\mathbf{y}) \ \ \forall\, \mathbf{y} \in \Omega.$$

Let $q \in \Omega$ be arbitrary, since Ω is open and connected, we find balls $B_{r_i}(\mathbf{x}_i)$ such that:

$$\mathbf{x}_j \in B_{r_{j-1}}(\mathbf{x}_{j-1}), \text{ for } j = 1, 2, \cdots, N, \ \mathbf{x}_0 = \mathbf{p}, \ \text{ and } \ \mathbf{x}_N = \mathbf{q}.$$

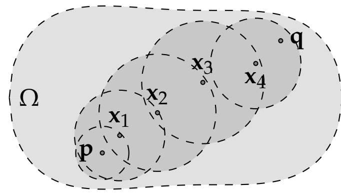

Figure 5.1. Using Mean Value Property to Construct Connected Balls.

Then, we need to prove that for each ball, its interior must have the same value with $\mathbf{p}$, then the sequence of the connected balls must shared the same value.
Here, note that with the mean value property, we assume that there exists $\mathbf{z} \in B_r(\mathbf{p})$ such that $u(\mathbf{z}) > m$, then there must exists another open ball $B_\delta(\mathbf{z}) \subset B_r(\mathbf{p})$, so we have:

$$\begin{aligned} m = \fint_{B_r(\mathbf{p})} u(\mathbf{y})d\mathbf{y} &= \frac{1}{|B_r(\mathbf{p})|}\left(\int_{B_r(\mathbf{p})\setminus B_\delta(\mathbf{z})} u(\mathbf{y})d\mathbf{y} + \int_{B_\delta(\mathbf{z})} u(\mathbf{y})d\mathbf{y}\right) \\ &= \frac{1}{|B_r(\mathbf{p})|}\left(\int_{B_r(\mathbf{p})\setminus B_\delta(\mathbf{z})} u(\mathbf{y})d\mathbf{y} + u(\mathbf{z})|B_\delta(\mathbf{z})|\right) \\ &> \frac{1}{|B_r(\mathbf{p})|}\left(\int_{B_r(\mathbf{p})\setminus B_\delta(\mathbf{z})} u(\mathbf{y})d\mathbf{y} + m|B_\delta(\mathbf{z})|\right) \\ &\geq \frac{1}{|B_r(\mathbf{p})|}\left(m|B_r(\mathbf{p}) \setminus B_\delta(\mathbf{z})| + m|B_\delta(\mathbf{z})|\right) = m. \end{aligned}$$

Since $m > m$ is impossible, this is a contradiction, so we must have that for all $\mathbf{x} \in B_r(\mathbf{z})$ that $u(\mathbf{x}) = u(\mathbf{p})$. Then, recursively constructing the equality of each circle, we know that for each point $\mathbf{q} \in \Omega$, we have $u(\mathbf{q}) = u(\mathbf{p})$.
The maximum $\mathbf{p}$ follows the same logic identically with it being minimum.

Remark 5.2.6. Hessian Matrix for Maximum Principle.

The *maximum principle* can also be proven by the second derivative test. Here, we construct the auxiliary function as:

$$V(x,y) = u(x,y) + \frac{M-m}{4R^2}[(x-x_0)^2 + (y-y_0)^2].$$

Assume that $u(x,y)$ attains its maximum value M at $(x_0, y_0) \in \Omega$ and $\max_{\partial\Omega} u(x,y) = m < M$. Since Ω is open and bounded, we let $R \in \mathbb{R}^+$ be a ball such that $B_R(\mathbf{0}) \supseteq \Omega$. Thus, for function:

$$V(x,y) = u(x,y) + \frac{M-m}{4R^2}[(x-x_0)^2 + (y-y_0)^2],$$

in which we now focus on the evaluation of this equation with the following diagram:

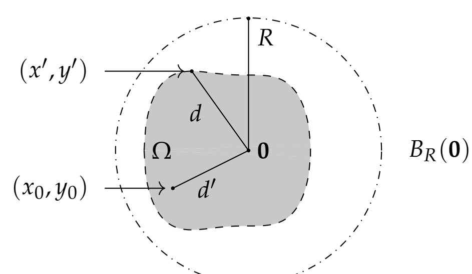

Figure 5.2. Triangle Inequality in Bounding $|(x', y') - (x_0, y_0)|$.

From the above demonstration, we notice that for any $(x', y') \in \partial\overline{\Omega}$, we have that $d := |(x', y')| \leq R$, and with $d' := |(x_0, y_0)| < R$ by triangle inequality, we have:

$$|(x', y') - (x_0, y_0)| < 2R,$$

hence we then know that:

$$\frac{M-m}{4R^2}|(x' - x_0, y' - y_0)|^2 < M - m \implies V(x', y') < m + M - m < M \text{ for all } (x', y') \in \partial\overline{\Omega}.$$

Note that we have for (x_0, y_0) that:

$$V(x_0, y_0) = u(x, y) = M,$$

which implies that $V(x, y)$ may not obtain its maximum value on $\partial\overline{\Omega}$, hence $V(x, y)$ obtain its maximum at $(\overline{\Omega})^\circ = \Omega$ since Ω is open.

Then, we evaluate the Hessian of $V(x, y)$, let $\mathbf{h} = (h_1, h_2)$ which is:

$$\begin{aligned} HV(x,y)(\mathbf{h}) &= \frac{1}{2}\begin{pmatrix} h_1 & h_2 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial^2 V}{\partial^2 x}(x,y) & \dfrac{\partial^2 V}{\partial x \partial y}(x,y) \\ \dfrac{\partial^2 V}{\partial y \partial x}(x,y) & \dfrac{\partial^2 V}{\partial^2 y}(x,y) \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} h_1 & h_2 \end{pmatrix} \cdot \begin{pmatrix} u_{xx} + \dfrac{M-m}{2R^2} & u_{xy} \\ u_{yx} & u_{yy} + \dfrac{M-m}{2R^2} \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \end{aligned}$$

Here, we evaluate the Hessian matrix, which is:

$$B = \begin{pmatrix} u_{xx} + \dfrac{M-m}{2R^2} & u_{xy} \\ u_{yx} & u_{yy} + \dfrac{M-m}{2R^2} \end{pmatrix}.$$

Since our assumption has shown that $V(x, y)$ obtains its maximum at Ω, then there must exist a point so that B is negative-definite, and by the Determinant Test, we must have:

$$u_{xx} + \frac{M-m}{2R^2} < 0 \text{ and } \left(u_{xx} + \frac{M-m}{2R^2}\right) \cdot \left(u_{yy} + \frac{M-m}{2R^2}\right) - u_{xy} \cdot u_{yx} > 0.$$

When we transfer the first condition, we have:

$$u_{xx} < -\frac{M-m}{2R^2} < 0.$$

For the right condition, and by Laplace equation, we have $u_{xx} + u_{yy} = 0$ we then have:

$$\begin{aligned} \left(u_{xx} + \frac{M-m}{2R^2}\right) \cdot \left(u_{yy} + \frac{M-m}{2R^2}\right) - u_{xy} \cdot u_{yx} &= u_{xx}u_{yy} + \frac{M-m}{2R^2}(u_{xx} + u_{yy}) + \left(\frac{M-m}{2R^2}\right)^2 - u_{xy}^2 \\ &= -u_{xx}^2 + \left(\frac{M-m}{2R^2}\right)^2 - u_{xy}^2 \\ &\leq -u_{xx}^2 + u_{xx}^2 - u_{xy}^2 = -u_{xy}^2 \leq 0, \end{aligned}$$

which is a contradiction, so by the second derivative test, we cannot obtain local maximum in the interior, which Ω, which is a contradiction to our assumption that $u(x, y)$ attains its maximum value at $(x_0, y_0) \in \Omega$, which implies that:

$$u(x, y) < \max_{\partial\Omega} u(x, y).$$

Overall, the *maximum property* embodies some of the most important underlying property for the *Laplace equation*. In the next sections, we will be inquiring about some special cases for the *Laplace equation*.

5.3 Poisson's Formula and Laplace Equation on a Disk

The first case concerns the Laplace Equation on a (open) disk.

Theorem 5.3.1. Poisson's Formula as Unique Solution.

Let the *Laplace equation* on a disk be formulated as:

$$\begin{cases} \Delta u(\mathbf{x}) = 0, & \text{for } \mathbf{x} \in B_R(\mathbf{p}); \\ u(\mathbf{x}) = g(\mathbf{x}), & \text{for } \mathbf{x} \in \partial B_R(\mathbf{p}). \end{cases}$$

The unique solution $u \in C^2(B_R) \cap C(\overline{B_R})$, given by *Poisson's formula*, is:

$$u(\mathbf{x}) = \frac{R^2 - |\mathbf{x} - \mathbf{p}|^2}{2\pi R} \int_{\partial B_R(\mathbf{p})} \frac{g(\boldsymbol{\sigma})}{|\mathbf{x} - \boldsymbol{\sigma}|^2} d\boldsymbol{\sigma}.$$

In particular, we have $u \in C^\infty(B_R)$.

For the case of a ball, one can naturally consider about the radial symmetry. Hence, we use the polar coordinates, that is:

$$x_1 = p_1 + r\cos\theta \text{ and } x_2 = p_2 + r\sin\theta.$$

Here, we replace u and g, respectively, into:

$$U(r,\theta) = u(p_1 + r\cos\theta, p_2 + r\sin\theta) \text{ and } G(\theta) = g(p_1 + R\cos\theta, p_2 + R\sin\theta).$$

Then, our *Laplace equation* becomes that:

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0, \text{ for } 0 < r < R,\ 0 \le \theta \le 2\pi,$$

with the condition becoming:

$$U(R,\theta) = G(\theta), \text{ for } 0 \le \theta \le 2\pi.$$

At the same time, the continuity guarantees that U and G are continuous on $[0,R] \times [0,2\pi]$ and $[0,2\pi]$, respective. Moreover, we must guarantee that a periodicity of 2π holds with respect to θ.
Then, we use the method of separation to separate U into:

$$U(r,\theta) = h(r)\phi(\theta).$$

With this substitution, we have:

$$h''(r)\phi(\theta) + \frac{1}{r}h'(r)\phi(\theta) + \frac{1}{r^2}h(r)\phi(\theta) = 0,$$

which further separates, and assign it to a constant value λ:

$$-\frac{r^2h''(r) + rh'(r)}{h(r)} = \frac{\phi''(\theta)}{\phi(\theta)} = -\lambda,$$

which we have a eigenvalue problem for $\phi(\theta)$, which is:

$$\begin{cases} \phi''(\theta) + \lambda\phi(\theta) = 0, \\ \phi(0) = \phi(2\pi). \end{cases}$$

Note that if $\lambda < 0$, the solution would be the linear combination of exponential functions, and by substituting in the boundary conditions, we would have the trivial solution, which is not desired. Hence, we must have $\lambda \ge 0$, and we have:

$$\phi(\theta) = \begin{cases} A + B\theta, & \lambda \neq 0, \\ A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta), & \lambda > 0. \end{cases}$$

With the 2π periodicity, it forces that $\lambda = n^2$, where $n \in \mathbb{N}$, or more specifically:

$$\phi_n(\theta) = \begin{cases} a, & n = 0, \\ a\cos(n\theta) + b\sin(n\theta), & n \geq 1. \end{cases}$$

Now, we consider the solution to $h(r)$, since we have $\lambda = n^2$, which is:

$$h_n(r) = \begin{cases} c\log r + d, & n = 0, \\ cr^{-n} + dr^n & n \geq 1. \end{cases}$$

Since h is harmonic, it cannot be unbounded on $[0, R]$, then we must have $c = 0$, so the system becomes:

$$h_n(r) = \begin{cases} d, & n = 0, \\ dr^n & n \geq 1. \end{cases}$$

Hence, we can form a collection of solutions:

$$U_n(r, \theta) = r^n\big(a_n\cos(n\theta) + b_n\sin(n\theta)\big),$$

which, by the principle of superposition, gives:

$$U(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n\big(a_n\cos(n\theta) + b_n\sin(n\theta)\big).$$

Here, a_n and b_n are the coefficients to be chosen in order to satisfy the boundary condition that:

$$\lim_{(r,\theta)\to(R,\xi)} U(r, \theta) = G(\xi), \text{ for all } \xi \in [0, 2\pi].$$

Here, we consider that:

$$G(\xi) = \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} \left[\alpha_m\cos(m\xi) + \beta_m\sin(m\xi)\right],$$

where:

$$\alpha_m = \frac{1}{\pi}\int_0^{2\pi} G(s)\cos(ms)ds \text{ and } \beta_m = \frac{1}{\pi}\int_0^{2\pi} G(s)\sin(ms)ds.$$

Therefore, we naturally assign by:

$$a_0 = \frac{\alpha_0}{2},\ a_n = R^{-m}\alpha_m, \text{ and } b_n = R^{-m}\beta_m.$$

As soon as we plug in the data back to our equation, we obtain that:

$$\begin{aligned} U(r, \theta) &= \frac{\alpha_0}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^m \int_0^{2\pi} G(s)\big[\cos(ns)\cos(n\theta) + \sin(ns)\sin(n\theta)\big]ds \\ &= \frac{1}{\pi}\int_0^{2\pi} G(s)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^m \cos\big(n(s-\theta)\big)\right] = \frac{1}{\pi}\int_0^{2\pi} G(s)\left[-\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^m \cos\big(n(s-\theta)\big)\right]. \end{aligned}$$

Remark 5.3.2. Uniform Convergence and the Poisson Formula.

When changing the order of sum with integration, we have assumed uniform convergence, which strictly restricts that $u \in C^1([0, 2\pi])$. However, even if we only have $G \in C([0, 2\pi])$, the convergence still holds.

Continuing with the proof, we want a better format of the some, so we use the *Euler's identity* that:

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n \cos\big(n(s-\theta)\big) &= \mathfrak{Re}\left[\sum_{n=1}^{\infty}\left(e^{i(s-\theta)}\frac{r}{R}\right)^n\right] \\ &= \mathfrak{Re}\frac{1}{1 - e^{i(s-\theta)}\cdot r/R} = \frac{R^2 - rR\cos(s-\theta)}{R^2 + r^2 - 2rR\cos(s-\theta)}. \end{aligned}$$

By subtracting an additional $\frac{1}{2}$, we have:

$$-\frac{1}{2}+\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n\cos\left(n(s-\theta)\right)=\frac{1}{2}\cdot\frac{R^2-r^2}{R^2+r^2-2rR\cos(s-\theta)},$$

and by inserting back to the integral, we can obtain that:

$$U(r,\theta)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{G(s)}{R^2+r^2-2Rr\cos(\theta-s)}ds.$$

Eventually, we convert back to the Cartesian coordinates, notice that:

$$\boldsymbol{\sigma}=\mathbf{p}+R(\cos s,\sin s),\ d\sigma=Rds,\ \text{ and } |\mathbf{x}-\boldsymbol{\sigma}|^2=R^2+r^2-2Rr\cos(\theta-s),$$

we have the formula as:

$$u(\mathbf{x})=\frac{R^2-|\mathbf{x}-\mathbf{p}|^2}{2\pi R}\int_{\partial B_R(\mathbf{p})}\frac{g(\boldsymbol{\sigma})}{|x-\boldsymbol{\sigma}|^2}d\sigma,$$

as desired. And by the Dirichlet condition, the solution is unique.

Meanwhile, for the case of a disk, we have the following proposition as an immediate result.

Proposition 5.3.3. Mean Value Property on Disk.

Let u be a harmonic function in a disk B, continuous on its closure $\overline{B}$. Then, the value of u at the center of D equals to the average of u con its circumference.

The justification can be done without loss of generality that $\mathbf{p}=\mathbf{0}$, then we have:

$$u(\mathbf{0})=\frac{R^2}{2\pi R}\int_{\partial B_R(\mathbf{0})}\frac{g(\boldsymbol{\sigma})}{R^2}d\sigma.$$

At the same time, we can extend the Poisson formula to all finite dimensional Euclidean space, namely as follows.

Proposition 5.3.4. Poisson's Formula for Disks in Finite Dimensional Euclidean Space.

When $B_R(\mathbf{p})$ is an d-dimensional ball, the solution of the Dirichlet problem is:

$$u(\mathbf{x})=\frac{R^2-|\mathbf{x}-\mathbf{p}|^2}{S(\partial B_1(\mathbf{0}))R}\int_{\partial B_R(\mathbf{p})}\frac{g(\boldsymbol{\sigma})}{|x-\boldsymbol{\sigma}|^d}d\sigma.$$

The result of the higher finite dimensional Euclidean space allows us to deduce another maximum principle.

Theorem 5.3.5. Harnack's Inequality.

Let u be harmonic and non-negative in a domain $\Omega\subset\mathbb{R}^n$. Assume that $B_R(\mathbf{z})\subset\subset\Omega$. Then, for any $\mathbf{x}\in\overline{B_R(\mathbf{z})}$, we have:

$$\frac{R^{n-2}(R-r)}{(R+r)^{n-1}}u(\mathbf{z})\le u(\mathbf{x})\le\frac{R^{n-2}(R+r)}{(R-r)^{n-1}}u(\mathbf{z}),$$

where $r=|\mathbf{x}-\mathbf{z}|$.

As a consequence, for every compact set $K\subset\Omega$, there exists a constant C, depending only on n, K and

the distance of K from $\partial\Omega$, such that:
$$\max_K(u) \leq C \min_K(u).$$

Without loss of generality, assume that $\mathbf{z} = \mathbf{0}$ and $\mathbf{0} \in \Omega$. Then, by the generalized Poisson's formula, we have that:
$$u(\mathbf{x}) = \frac{R^2 - |\mathbf{x}|^2}{S(\partial B_1(\mathbf{0}))R} \int_{\partial B_R(\mathbf{0})} \frac{u(\sigma)}{|\sigma - \mathbf{x}|^n} d\sigma.$$
With *Triangle Inequality*, we have demonstrated that:
$$R - |\mathbf{x}| \leq |\sigma - \mathbf{x}| \leq R + |\mathbf{x}|,$$
and observe that:
$$R^2 - |\mathbf{x}|^2 = (R - |\mathbf{x}|)(R + |\mathbf{x}|),$$
we can apply the mean value property that:
$$\begin{aligned} u(\mathbf{x}) &\leq \frac{R + |\mathbf{x}|}{(R - |\mathbf{x}|)^{n-1}} \cdot \frac{1}{S(\partial B_1(\mathbf{0}))R} \int_{\partial B_R(\mathbf{0})} u(\sigma) d\sigma \\ &= \frac{R^{n-1}(R - |\mathbf{x}|)}{(R + |\mathbf{x}|)^{n-1}} \cdot \frac{1}{S(\partial B_1(\mathbf{0}))R} \int_{\partial B_R(\mathbf{0})} u(\sigma) d\sigma = \frac{R^{n-2}(R + |\mathbf{x}|)}{(R - |\mathbf{x}|)^{n-1}} u(\mathbf{0}). \end{aligned}$$
The other inequality follows analogously.
For the final inequality, we can construct a sequence of balls from the minimum point to the maximum point, such that each balls are contained in Ω and the center of the latter ball is contained in the first.

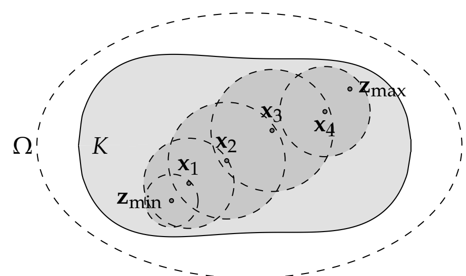

Figure 5.3. Construct a Sequence of Balls Connecting the Minimum and Maximum.

Since the number of ball are dependent on the distance between K and $\partial\Omega$. And with a given number of the balls, we can us the previous inequality to obtain a bound C.

5.4 Laplace Equation on Rectangle Region

Here, we consider the Laplace equation in $R = [0, L] \times [0, H]$, as follows:
$$\begin{cases} \text{P.D.E.: } \Delta u(x, y) = 0, & (x, y) \in R; \\ \text{B.C.: } \begin{cases} \text{B.C.1: } u(0, y) = g_1(y), & y \in [0, H]; \\ \text{B.C.2: } u(L, y) = g_2(y), & y \in [0, H]; \\ \text{B.C.3: } u(x, 0) = f_1(x), & x \in [0, L]; \\ \text{B.C.4: } u(x, H) = f_2(x), & x \in [0, L]. \end{cases} \end{cases}$$
The strategy for solving such problem is to use the *principle of superposition*, *i.e.*, subdividing the problem into 4 separated *Laplace equation*, each with one boundary condition and the other being set to 0.

Remark 5.4.1. Laplace Equation on Higher Dimensional Rectangles.

The Laplace Equation of Rectangle extends to higher dimensions:

(i) At dimension 2, our rectangle contains 4 boundary conditions, each representing an edge.

(ii) At dimension 3, the boundary conditions is consisted of 6 rectangle hyper-planes, namely the 6 “faces” of the block.

(iii) The higher dimensions follows the same pattern, *i.e.*, breaking up the boundary into hyper-planes.

Here, we give an example of solving the 2D Rectangle Region with one single line being non-trivial.

Example 5.4.2. Laplace Equation on 2D Rectangle Region with 1 Non-trivial Edge.

Likewise, we consider one of the Laplace equation in $R = [0, L] \times [0, H]$, as follows:

$$\begin{cases} \text{P.D.E.: } \Delta u(x,y) = 0, & (x,y) \in R; \\ \text{B.C.: } \begin{cases} \text{B.C.1: } u(0,y) = g_1(y), & y \in [0,H]; \\ \text{B.C.2: } u(L,y) = 0, & y \in [0,H]; \\ \text{B.C.3: } u(x,0) = 0, & x \in [0,L]; \\ \text{B.C.4: } u(x,H) = 0, & x \in [0,L]. \end{cases} \end{cases}$$

Here, we use the method of separation, let:

$$u(x,y) = h(x)\phi(y),$$

which pushes our three homogeneous boundary conditions into:

$$h(L) = \phi(0) = \phi(H) = 0.$$

Moreover, we have the Laplace equation substituted as:

$$\Delta\big(h(x)\phi(y)\big) = \phi(y)\frac{d^2h(x)}{dx^2} + h(x)\frac{d^2\phi(y)}{dy^2} = 0.$$

Then, we would separate the variables and assign them with a λ:

$$\frac{1}{h(x)}h''(x) = -\frac{1}{\phi(y)}\phi''(y) = \lambda,$$

which gives us two ordinary differential equations:

$$\begin{cases} h''(x) - \lambda h(x) = 0, \\ \phi''(y) + \lambda\phi(y) = 0. \end{cases}$$

Here, we have $\phi(y)$ with two homogeneous boundary conditions, and it is an eigenvalue problem with respect to y. Consider that if $\lambda \leq 0$, we would have the solution as linear combination of exponential function, and the initial conditions push the solution to be 0, or the trivial solution, which is undesired. Hence, we let $\lambda > 0$, then the solution is:

$$\phi(y) = A\sin(\sqrt{\lambda}y) + B\cos(\sqrt{\lambda}y).$$

Note that with the initial condition, $\phi(0) = 0$, it forces B to be zero, hence we now have:

$$\phi(y) = A\sin(\sqrt{\lambda}y),$$

but given that $\phi(H) = 0$, this further forces the eigenvalues be:

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2, \text{ for } n \in \mathbb{Z}^+,$$

and the eigenfunctions are:

$$\phi_n(y) = \sin\left(\frac{n\pi y}{H}\right), \text{ for } n \in \mathbb{Z}^+.$$

Now, we shift our attention back to $h(x)$, which at every $\lambda_n = \left(\frac{n\pi}{H}\right)^2$, giving that:

$$h_n''(x) = \left(\frac{n\pi}{H}\right)^2 h(x),$$

which we note that the following is a linearly independent solution:

$$h_n(x) = C\cosh\left(\frac{n\pi}{H}(x-L)\right) + D\sinh\left(\frac{n\pi}{H}(x-L)\right),$$

and here, with the initial condition of $h(L) = 0$, we have that $\cosh(0) = 1$, so $C = 0$, which we have the solution as:

$$u_n(x,y) = A_n \sin\left(\frac{n\pi y}{H}\right)\sinh\left(\frac{n\pi}{H}(x-L)\right),$$

which by principle of superposition, we have:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right)\sinh\left(\frac{n\pi}{H}(x-L)\right),$$

Eventually, we evaluate it at and $y = 0$, with condition $u(0,y) = g_1(y)$, it gives that:

$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right)\sinh\left(-\frac{nL\pi}{H}\right).$$

Here, we associates $A_n \sinh(-nL\pi/H)$ as its coefficients. By orthogonality, we have:

$$A_n \sinh\left(-\frac{nL\pi}{H}\right) = \frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right) dy.$$

Thus, eventually, we have that $\sinh(nL\pi/H)$ being non-zero, then we have the coefficient that:

$$A_n = \frac{2}{H\sinh(nL\pi/H)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right) dy.$$

Hence, we have obtained the solution satisfying one edge. Eventually, we superpose all solutions of all edges to obtain a solution for the whole rectangle.

5.5 Cauchy Problem for Laplace Equation

Prior to discussing the case in $\mathbb{R}^n$, we want to first recall some properties of the Laplace operator.

Proposition 5.5.1. Invariance Properties of the Laplace Operator Δ.

The Laplace operator follows the below properties:

(i) Δ is translation invariant, meaning that:

$$\Delta\big[u(\mathbf{x}-\mathbf{y})\big] = (\Delta u)(\mathbf{x}-\mathbf{y}).$$

Meanwhile, this property immediately causes that if $u(\mathbf{x})$ is harmonic in $\Omega \subset \mathbb{R}^n$, then $v(\mathbf{x}) = u(\mathbf{x}-\mathbf{y})$ is harmonic on $\Omega + \mathbf{y} := \{\mathbf{x}+\mathbf{y} : \mathbf{x} \in \Omega\}$.

(ii) Δ is rotation invariant, meaning that for any orthogonal matrix $\mathbf{M}$ (such that $\mathbf{M}^T = \mathbf{M}^{-1}$), we have:

$$\Delta\big[u(\mathbf{M}.\mathbf{x})\big] = (\Delta u)(\mathbf{M}.\mathbf{x}).$$

In particular, if $u(\mathbf{x})$ is harmonic in $\Omega \subset \mathbb{R}^n$, then $v(\mathbf{x}) = u(\mathbf{M}.\mathbf{x})$ is harmonic on $\mathbf{M}^{-1}.\Omega := \{\mathbf{M}^{-1}.\mathbf{x} : \mathbf{x} \in \Omega\}$.

Here, the translation invariant is almost immediate from the definition of Δ operator. For the translation invariant, we may use the trace to assist our proof:

$$\Delta u(\mathbf{M}.\mathbf{x}) = \text{Trace}\left(D^2 u(\mathbf{M}.\mathbf{x})\right) = \text{Trace}\left(\mathbf{M}^{-1}.D^2 u(\mathbf{M}.\mathbf{x}).\mathbf{M}\right) = \Delta\big[u(\mathbf{M}.\mathbf{x})\big].$$

The *Laplace operator* can be helpful in terms of the radially symmetry cases. Now we consider the fundamental solution for dimensions 2 and 3.

Theorem 5.5.2. Fundamental Solution of Laplace Operator in Dimension 2 and 3.

The fundamental solution for the Laplace operator Δ for dimensions 2 and 3 are:

$$\Phi(\mathbf{x}) = \begin{cases} -\dfrac{1}{2\pi}\log|\mathbf{x}|, & n = 2; \\ \dfrac{1}{4\pi|\mathbf{x}|}, & n = 3. \end{cases}$$

For the $n = 2$ case, we consider the polar coordinates, so we have:

$$u_{rr} + \frac{1}{r}u_r = 0,$$

which results in $u_r = C/r$, then $u(r) = C\log r + C_1$ for $C, C_1 \in \mathbb{R}$.
Likewise, in $n = 3$ case, we consider the spherical coordinates, where we have that:

$$\Delta = \underbrace{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}}_{\text{radial part}} + \frac{1}{r^2}\underbrace{\left[\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\phi^2} + \cot\phi\frac{\partial}{\partial\phi}\right]}_{\text{spherical part (Laplace-Beltrami operator)}}.$$

Then, the Laplace equation becomes:

$$u_{rr} + \frac{2}{r}u_r = 0,$$

which results in $u_r = \tilde{C}/r^2$, hence $u(r) = C/r + C_1$.
Here, we must choose that:

$$\begin{cases} C_1 = 0,\ C = -\dfrac{1}{2\pi}, & n = 2; \\ C_1 = 0,\ C = \dfrac{1}{4\pi}, & n = 3; \end{cases}$$

which leads to the fundamental solutions.

Remark 5.5.3. Fundamental Solution and Dirac Delta Function.

The above choice of C and C_1 are made in order that:

$$\Delta\Phi(\mathbf{x}) = -\delta_n(\mathbf{x}).$$

In such process, we use the weak convergence of:

$$\langle\Delta\Phi, \phi\rangle \overset{\Delta}{=} \langle\Phi, \Delta\phi\rangle = -\phi(\mathbf{0})$$

to show the convergence to the Dirac delta function δ_n.
In particular, we want to construct the region $\operatorname{supp}\phi \subset B_R(\mathbf{0})$ and with $\epsilon \to 0$ that:

$$\lim_{\epsilon\to 0}\int_{B_R(\mathbf{0})\setminus B_\epsilon(\mathbf{0})} \Phi(\mathbf{x})\Delta\phi(\mathbf{x})d\mathbf{x} = -\phi(\mathbf{0}).$$

At the same moment, we can also generalize the *fundamental solution* into higher finite dimensional Euclidean Space.

Proposition 5.5.4. Fundamental Solution for Dimensions Higher than 3.

In dimension $n > 3$, the equation for $u(r)$ is:

$$u_{rr} + \frac{n-1}{r} u_r = 0,$$

and the fundamental solution of the Laplace operator is:

$$\Phi(\mathbf{x}) = \frac{1}{(n-2)S(\partial B_1(\mathbf{0}))} \frac{1}{|\mathbf{x}|^{n-2}}.$$

The higher dimensional Laplace operator would depend on the specific coordinate system resembling each point by a length and all the rest as angles.

If we consider $f(\mathbf{x})/(4\pi)$ as the density of charge inside a compact set in $\mathbb{R}^3$, then $\Phi(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y}$ resembles the potential at $\mathbf{x}$ due to the charge $f(\mathbf{y})d\mathbf{y}/(4\pi)$ inside the region of volume $d\mathbf{y}$ around $\mathbf{y}$. Hence, the full potential as a concept of the Newtonian potential.

Definition 5.5.5. Newtonian Potential.

The *Newtonian potential* of f defined on $\mathbb{R}^3$ is:

$$u(\mathbf{x}) = (\Phi * f)(\mathbf{y}) = \int_{\mathbb{R}^3} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y}.$$

Note that by construction, we cannot directly apply the Δ operator on the Newtonian potential, as if we have $r = |\mathbf{x}|$, we have:

$$\Phi_{x_j x_j}(r) = -\frac{1}{r^3} + \frac{3x_j^2}{r^5},$$

which is not integrable in $\mathbb{R}^3$, and thus cannot be differentiated twice. We would need the construction removing the central part.

Theorem 5.5.6. Newtonian Potential as Solution to Function with Compact Support.

Let $f \in C^2(\mathbb{R}^3)$ be a function with compact support. Let u be the Newtonian potential of f, then u is the unique solution in $\mathbb{R}^3$ of:

$$\Delta u = -f$$

which belongs to $C^2(\mathbb{R}^3)$ and vanishes at infinity.

The vanishing at infinity can be thought of the compensation for the absence of boundary condition.

Remark 5.5.7. Newtonian Potential Holding in Dimension 2.

The appropriate version holds in dimension $n = 2$, namely by replacing into *logarithmic potential*, that is:

$$u(\mathbf{x}) = (\Phi * f)(\mathbf{y}) d\mathbf{y} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|\mathbf{x}-\mathbf{y}| f(\mathbf{y}) d\mathbf{y}.$$

If one evaluates the potential, it does not vanish at infinity, but accomplish the asymptotic behavior for $|x| \to \infty$ of:

$$u(\mathbf{x}) = -\frac{M}{2\pi} \log|\mathbf{x}| + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right),$$

where:

$$M = \int_{\mathbb{R}^2} f(\mathbf{y}) d\mathbf{y}.$$

5.6 Green's Function and Representation

When having a bounded domain, we cannot lift the conditions for boundary conditions, *i.e.*, any representation formula has to take into account the boundary values.

Theorem 5.6.1. Green's Relation with Newtonian Potential.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain and $u \in C^2(\overline{\Omega})$. Then, for every $\mathbf{x} \in \Omega$:

$$u(\mathbf{x}) = -\int_{\Omega} \Phi(\mathbf{x}-\mathbf{y}) \Delta u(\mathbf{y}) d\mathbf{y} + \int_{\partial\Omega} \Phi(\mathbf{x}-\sigma) \partial_{\mathbf{n}} u(\sigma) d\sigma - \int_{\partial\Omega} u(\sigma) \partial_{\mathbf{n}} \Phi(\mathbf{x}-\sigma) d\sigma.$$

On the right hand side, the first term is the Newtonian potential of $-\Delta u$, while the last two terms are the single and double layer potentials of $\partial_{\mathbf{n}} u$ and $-u$.

Again, we want to apply *Green's second identity* on u and $\Phi(\mathbf{x} - \bullet)$. However, since $\Phi(\mathbf{x} - \bullet)$ has singularity at $\mathbf{x}$, we construct the ball removing the singularity, say:

$$\Omega \setminus \overline{B_\epsilon(\mathbf{x})},$$

so that $\Phi(\mathbf{x} - \bullet)$ is smooth and harmonic in this region. Thus, we have:

$$\int_{\Omega \setminus \overline{B_\epsilon(\mathbf{x})}} \left[u(\mathbf{y}) \Delta \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}-\mathbf{y}) \Delta u(\mathbf{y}) \right] d\mathbf{y} = \int_{\partial(\Omega \setminus \overline{B_\epsilon(\mathbf{x})})} \left[u(\sigma) \partial_{\mathbf{n}} \Phi(\mathbf{x}-\sigma) - \Phi(\mathbf{x}-\sigma) \partial_{\mathbf{n}} u(\sigma) \right] d\sigma.$$

Note that since $\Delta_{\mathbf{y}} \Phi(\mathbf{x}-\mathbf{y}) = 0$, we have:

$$\int_{\Omega \setminus \overline{B_\epsilon(\mathbf{x})}} \left[-\Phi(\mathbf{x}-\mathbf{y}) \Delta u(\mathbf{y}) \right] d\mathbf{y} = \int_{\partial(\Omega \setminus \overline{B_\epsilon(\mathbf{x})})} \left[u(\sigma) \partial_{\mathbf{n}} \Phi(\mathbf{x}-\sigma) - \Phi(\mathbf{x}-\sigma) \partial_{\mathbf{n}} u(\sigma) \right] d\sigma.$$

In dimension 3, we replace by the fundamental solution, which gives us that:

$$\begin{aligned} \int_{\Omega \setminus \overline{B_\epsilon(\mathbf{x})}} \left[-\frac{\Delta u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right] d\mathbf{y} &= \int_{\partial(\Omega \setminus \overline{B_\epsilon(\mathbf{x})})} \left[u(\sigma) \partial_{\mathbf{n}} \frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|} \partial_{\mathbf{n}} u(\sigma) \right] d\sigma \\ &= \int_{\partial\Omega \cup \partial B_\epsilon(\mathbf{x})} \left[u(\sigma) \partial_{\mathbf{n}} \frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|} \partial_{\mathbf{n}} u(\sigma) \right] d\sigma. \end{aligned}$$

As $\epsilon \to 0$, by the *dominated convergence theorem*, the left hand side converges as:

$$\int_{\Omega \setminus \overline{B_\epsilon(\mathbf{x})}} \left[-\frac{\Delta u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right] d\mathbf{y} \to \int_{\Omega} \chi_{\Omega \setminus \overline{B_\epsilon(\mathbf{x})}}(\mathbf{y}) \left[-\frac{\Delta u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right] d\mathbf{y} \to -\int_{\Omega} \left[\frac{\Delta u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right] d\mathbf{y}$$

For the two terms with $\partial B_\epsilon(\mathbf{x})$, they converge, respectively to:

$$0 \text{ and } u(\mathbf{x}),$$

which plugging them back to the equation gives the Green's Relation.

Then, we introduce the Green's function as a solution for the Δ operator.

Definition 5.6.2. Green's Function.

For fixed $\mathbf{x} \in \Omega$, G satisfies that:

$$\begin{cases} \Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) = -\delta_3(\mathbf{x}-\mathbf{y}) & \text{for } \mathbf{y} \in \Omega; \\ G(\mathbf{x}, \sigma) = 0, & \text{for } \sigma \in \partial\Omega. \end{cases}$$

Explicitly, we can write G as:

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \varphi(\mathbf{x},\mathbf{y}),$$

where φ solves the following Dirichlet problem:

$$\begin{cases} \Delta_{\mathbf{y}}\varphi(\mathbf{x},\mathbf{y}) = 0, & \text{for } \mathbf{y} \in \Omega; \\ \varphi(\mathbf{x},\sigma) = \Phi(\mathbf{x}-\sigma), & \text{for } \mathbf{y} \in \partial\Omega; \end{cases}$$

for fixed $\mathbf{x} \in \Omega$.

At the same time, the Green's Formula satisfies some important properties.

Proposition 5.6.3. Properties of the Green's Formula.

The Green function has the following properties:

(i) Positivity: $G(\mathbf{x},\mathbf{y}) > 0$ for every $\mathbf{x},\mathbf{y} \in \Omega$, with $G(\mathbf{x}-\mathbf{y}) \to +\infty$ when $\mathbf{x}-\mathbf{y} \to \mathbf{0}$;

(ii) Symmetry: $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$.

The Green's formula presents good evidence to existence, but it only gives the explicit formula for specific domains, such as using the Poisson's formula for open balls.

Remark 5.6.4. Green's Formula as Definition.

Here, we can adapt the notation that:

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \varphi(\mathbf{x},\mathbf{y})$$

as the definition function for the Laplace operator in a domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$.

In particular, we have the *method of electrostatic images* attempting to expand the formulation for a domain.

Example 5.6.5. Method of Electrostatic Images.

We start with the Green's function for the sphere, namely for $\Omega = B_R(\mathbf{0}) \subset \mathbb{R}^3$. To determine the Green's function, we set:

$$\varphi(\mathbf{x},\mathbf{y}) = \frac{q}{4\pi|\mathbf{x}^* - \mathbf{y}|},$$

where $\mathbf{x}$ is fixed in Ω, q is an imaginary charge placed at a suitable point $\mathbf{x}^*$.

Now, we attempt to determine $\mathbf{x}^*$, as it must satisfy that:

$$\frac{q}{|\mathbf{x}^* - \mathbf{y}|} = \frac{1}{|\mathbf{x}-\mathbf{y}|},$$

when $|\mathbf{y}| = R$, which yields that:

$$|\mathbf{x}^* - \mathbf{y}|^2 = q^2|\mathbf{x}-\mathbf{y}|^2,$$

or specifically as:

$$|\mathbf{x}^*|^2 - 2\mathbf{x}^* \cdot \mathbf{y} + R^2 = q^2(|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{y} + R^2).$$

When we rearrange the terms, we have:

$$|\mathbf{x}^*|^2 + R^2 - q^2(R^2 + |\mathbf{x}|^2) = 2\mathbf{y}\cdot(\mathbf{x}^* - q^2\mathbf{x}).$$

Note that the left hand side does not depend on $\mathbf{y}$, this forces both sides to be zero, then we have:

$$\mathbf{x}^* = q^2\mathbf{x},$$

and so we have:

$$q^4|\mathbf{x}|^2 - q^2(R^2 + |\mathbf{x}|^2) + R^2 = 0,$$

by which yielding that:

$$q = \frac{R}{|\mathbf{x}|}.$$

In particular, the image that we have corresponds as:

$$\mathbf{x}^* = \frac{R^2}{|\mathbf{x}|^2}\mathbf{x}.$$

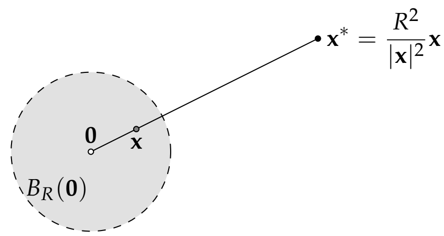

Figure 5.4. Image $\mathbf{x}^*$ Constructed from $\mathbf{x}$ using Imaginary Charge.

Hence, for any $\mathbf{x} \neq 0$, it gives that:

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi}\left[\frac{1}{|\mathbf{x} - \mathbf{y}|} - \frac{R}{|\mathbf{x}||\mathbf{x}^* - \mathbf{y}|}\right].$$

Then, we want to find a way to define G on $\mathbf{x} = \mathbf{0}$. Here, we attempt to let $\mathbf{x} \to \mathbf{0}$, we have that:

$$\varphi(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi}\frac{R}{|\mathbf{x}||\mathbf{x}^* - \mathbf{y}|} \to \frac{1}{4\pi R}.$$

Hence, it is justified for us to define that:

$$G(\mathbf{0}, \mathbf{y}) = \frac{1}{4\pi}\left[\frac{1}{|\mathbf{y}|} - \frac{1}{R}\right].$$

Now, we have defined that:

$$G(\mathbf{x}, \mathbf{y}) = \begin{cases} \Phi(\mathbf{x} - \mathbf{y}) - \Phi\left(\frac{R}{|\mathbf{x}|}|\mathbf{x} - \mathbf{y}|\right), & \text{for } \mathbf{x} \neq \mathbf{0}; \\ \Phi(\mathbf{y}) - \Phi(R\mathbf{n}), & \mathbf{x} = \mathbf{0}. \end{cases}$$