AS.110.653 : Stochastic Differential Equations

Practice Problem Sets

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Contents

- The pratice problem sets are practices for AS.110.653 Stochastic Differential Equationsinstructed by *Dr. Xiong Wang* at *Johns Hopkins University* in the Spring 2025 semester.
  - Dr. Wang has really dedicated a lot into designing and executing the class. We greatly appreciate his instructions throughout the course and his assistance in tackling on these problems.
- Exercises numbers refer to the course textbook [Øksendal]:
  - *Stochastic Differential Equations: An Introduction with Applications* by Bernt Øksendal.
- The solutions might contain minor typos or errors. Please point out any notable error(s) through this link.

I Problem Set 1

Problem I.1. (Exercise 2.1 on [Øksendal]). Suppose that $X : \Omega \to \mathbb{R}$ is a function which assumes only countably many values $a_1, a_2, \cdots \in \mathbb{R}$.

(a) Show that X is a random variable if and only if:

$$X^{-1}(a_k) \in \mathcal{F} \text{ for all } k = 1, 2, \cdots. \tag{1}$$

Proof. Here, note that X assumes only countably many values $a_1, a_2, \cdots \in \mathbb{R}$, and denote the set of these points as $X(\Omega)$, for any open set $U \subset \mathbb{R}$, its preimage $X^{-1}(U)$ must be a subset of $X(\Omega)$, *i.e.*, $X^{-1}(U) \subset X^{-1}\big(X(\Omega)\big)$. Now, let $I \subset \mathbb{N}^+$ be a indexed set in which $a_i \in U$, then the preimage of U is simply the countable union $X^{-1}(U) = \bigcup_{i \in I} X^{-1}(a_i)$.
Recall that for a σ-algebra, if a sequence of set is in it, its countable union must be still in it. Note that $X(\Omega)$ is countable, it is discrete (or not containing an interval in $\mathbb{R}$, which making it uncountable), so for any a_j where $j \in \mathbb{N}^+$, there exists some $\epsilon > 0$ such that $a_k \notin N_\epsilon(a_j)$ for all $k \neq j$. By such, we know that X being a random variable is equivalent to saying that $X^{-1}(U) \in \mathcal{F}$ for all open set $U \subset \mathbb{R}$, which is equivalent to saying that $X^{-1}(\bigcup_{i \in I}) a_i \in \mathcal{F}$ for all possible $I \in \mathcal{P}(\mathbb{N}^+)$, which is equivalently to $X^{-1}(a_k) \in \mathbb{F}$ for all $k \in \mathbb{Z}^+$, as desired. □

(b) Suppose (1) holds, show that:

$$\mathbb{E}[|X|] = \sum_{k=1}^{\infty} |a_k| \mathbb{P}[X = a_k]$$

Proof. Now, as we shall evaluate the expectation, while $X(\Omega)$ is countable, we have:

$$\begin{aligned} \mathbb{E}[|X|] &= \int_\Omega |X(\omega)| \, d\mathbb{P}(\omega) = \int_{X(\Omega)} |a| \, d\mathbb{P}\big(X^{-1}(a)\big) \\ &= \sum_{a \in X(\Omega)} |a| \mathbb{P}\big(X^{-1}(a)\big) = \sum_{k=1}^{\infty} |a_k| \mathbb{P}[X = a_k], \end{aligned}$$

as desired. □

(c) If (1) holds and $\mathbb{E}[|X|] < \infty$, show that:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} a_k \mathbb{P}[X = a_k].$$

Proof. By (1) and $\mathbb{E}[|X|] < \infty$, we know that $|X(\omega)|$ is integrable, then, we may evaluate the integral without the absolute value sign (which is not necessarily positive):

$$\begin{aligned} \mathbb{E}[X] &= \int_\Omega X(\omega) \, d\mathbb{P}(\omega) = \int_{X(\Omega)} a \, d\mathbb{P}\big(X^{-1}(a)\big) \\ &= \sum_{a \in X(\Omega)} a \mathbb{P}\big(X^{-1}(a)\big) = \sum_{k=1}^{\infty} a_k \mathbb{P}[X = a_k]. \end{aligned}$$

Note that based on the definition of Lebesgue integration, a function is integrated on the positive and negative parts, respectively, so we must enforce convergence in absolute value (absolute convergence) for the integral to be well defined. □

(d) If (1) holds and $f : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, show that:

$$\mathbb{E}[f(X)] = \sum_{k=1}^{\infty} f(a_k)\mathbb{P}[X = a_k].$$

Proof. First, we need to show that $\mathbb{E}[|f(X)|]$ is finite. Since f is bounded, there exists some $C \in \mathbb{R}^+$ such that $|f(x)| < C$ for all $x \in \mathbb{R}$. Moreover, since f is measurable, and $X(\Omega)$ is discrete, then $f\big(X(\Omega)\big)$ is discrete (thus measurable) and for any $x \in f\big(X(\Omega)\big)$, $f^{-1}(x)$ is measurable, hence, we have the expectation as:

$$\begin{aligned}
\mathbb{E}[|f(X)|] &= \int_{\Omega} \left|f\big(X(\omega)\big)\right| \, d\mathbb{P}(\omega) = \int_{X(\Omega)} |f(a)| \, d\mathbb{P}\big(X^{-1}(a)\big) \\
&= \sum_{a \in X(\Omega)} |f(a)| \mathbb{P}\big(X^{-1}(a)\big) = \sum_{k=1}^{\infty} |a_k| \mathbb{P}[X = a_k] \\
&< C \sum_{k=1}^{\infty} \mathbb{P}[X = a_k] = C < \infty.
\end{aligned}$$

Hence, it is integrable, so we may find the expectation without absolute value sign, that is:

$$\begin{aligned}
\mathbb{E}[f(X)] &= \int_{\Omega} f\big(X(\omega)\big) \, d\mathbb{P}(\omega) = \int_{X(\Omega)} f(a) \, d\mathbb{P}\big(X^{-1}(a)\big) \\
&= \sum_{a \in X(\Omega)} f(a) \mathbb{P}\big(X^{-1}(a)\big) = \sum_{k=1}^{\infty} f(a_k) \mathbb{P}[X = a_k],
\end{aligned}$$

which finishes the proof. □

Problem I.2. (Exercise 2.3 on [Øksendal]). Let $\{\mathcal{H}_i\}_{i \in I}$ be a family of σ-algebras on Ω. Prove that:

$$\mathcal{H} = \bigcap \{\mathcal{H}_i : i \in I\}$$

is again a σ-algebra.

Proof. First, we note that each σ-algebra contains $\emptyset$, hence their intersection shall still contain $\emptyset$.
Now, for any $F \in \mathcal{H}$, we know that $F \in \mathcal{H}_i$ for all $i \in I$, then $F^c \in \mathcal{H}_i$ for all $i \in I$, thus $F^c \in \mathcal{H}$.
Eventually, let $\{F_a\}_{a \in \mathbb{N}^+} \subset \mathcal{H}$ be an arbitrary sequence, then $\{F_a\}_{a \in \mathbb{N}^+} \subset \mathcal{H}_i$ for all $i \in I$, then $\bigcup_{a \in \mathbb{N}^+} F_a \in \mathcal{H}_i$ for all $i \in I$, hence the countable union is in $\mathcal{H}$.
Thus, $\mathcal{H}$ is a σ-algebra. □

Problem I.3. (Exercise 2.4 in [Øksendal]).

(a) Let $X : \Omega \to \mathbb{R}^n$ be a random variable such that:

$$\mathbb{E}[|X|^p] < \infty \text{ for some } p, 0 < p < \infty.$$

Prove *Chebychev's inequality*:

$$\mathbb{P}[|X| \geq \lambda] \leq \frac{1}{\lambda^p}\mathbb{E}[|X|^p] \text{ for all } \lambda > 0.$$

Hint: $\int_\Omega |X|^p \, d\mathbb{P} \geq \int_A |X|^p \, d\mathbb{P}$, where $A = \{\omega : |X| \geq \lambda\}$.

Proof. Here, we first note that $A \subset \Omega$, so we trivially have:

$$\int_\Omega |X|^p d\mathbb{P} \geq \int_A |X|^p \, d\mathbb{P},$$

by the monotonicity measure of subsets.
Then, we may build an inequality as:

$$\begin{aligned}\int_\Omega |X|^p \, d\mathbb{P} \geq \int_A |X|^p \, d\mathbb{P} &= \int_A |X(\omega)|^p \, d\mathbb{P}(\omega) \\ &\geq \int_A \lambda^p \, d\mathbb{P}(\omega) = \lambda^p \int_A d\mathbb{P}(\omega) = \lambda^p \mathbb{P}(A) = \lambda^p \mathbb{P}[|X| \geq \lambda].\end{aligned}$$

Then, by dividing both sides with λ^p, we now have:

$$\mathbb{P}[|X| \geq \lambda] \leq \frac{1}{\lambda^p} \int_\Omega |X|^p \, d\mathbb{P} = \frac{1}{\lambda^p}\mathbb{E}[|X|^p],$$

which completes the proof. □

(b) Suppose there exists $k > 0$ such that:

$$M = \mathbb{E}[\exp(k|X|)] < \infty.$$

Prove that $\mathbb{P}[|X| \geq \lambda] \leq Me^{-k\lambda}$ for all $\lambda \geq 0$.

Proof. Here, can first note that since $\exp(-)$ is monotonic, so:

$$\mathbb{P}[|X| \geq \lambda] = \mathbb{P}[|\exp(k|X|)| \geq e^{k\lambda}].$$

Since we assume that $M = \mathbb{E}[\exp(k|X|)] < \infty$, we can apply part (a) with $p = 1$ as:

$$\mathbb{P}[|\exp(k|X|)| \geq e^{k\lambda}] \leq \frac{1}{e^{k\lambda}}\mathbb{E}[|\exp(k|X|)|] = \frac{1}{e^{k\lambda}}\mathbb{E}[\exp(k|X|)] = Me^{-k\lambda},$$

and it combines with the previous equality as:

$$\mathbb{P}[|X| \geq \lambda] \leq Me^{-k\lambda},$$

as desired. □

Problem I.4. (Exercise 2.6 in [Øksendal]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A_1, A_2, \cdots$ be sets in $\mathcal{F}$ such that:

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty.$$

Prove the *Borel-Cantelli* lemma:

$$\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) = 0,$$

i.e., the probability that ω belongs to infinitely many A_k's is zero.

Proof. First, we note that $\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k$ is a countable intersection of countable union of measurable set, hence $\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k \in \mathcal{F}$, *i.e.*it is measurable.
Then, note that the infinite sum $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, then for any $\epsilon > 0$, there exists some $m > 0$ such that:

$$\sum_{k=m}^{\infty} \mathbb{P}(A_k) < \epsilon.$$

Thus, we can note that by the fact that an intersection is a subset and by the countable additivity of measure, we have:

$$\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) \leq \mathbb{P}\left(\bigcup_{k=m}^{\infty} A_k\right) < \epsilon.$$

Now, since $\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) < \epsilon$ for all $\epsilon > 0$, we have:

$$\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) = 0,$$

which completes the proof of the *Borel-Cantelli* lemma. □

Problem I.5. Prove Lebesgue's dominance convergence theorem under assumption "convergence in probability." You can apply the version under assumption "convergence almost surely."

Here, we first recall Lebesgue's dominance convergence theorem:

Theorem. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ for a.e. x, as $n \to \infty$. If $|f_n(x)| \leq g(x)$, where g is integrable, then:

$$\int |f_n - f| \to 0 \text{ as } n \to \infty,$$

and consequently:

$$\int f_n \to \int f \text{ as } n \to \infty.$$

To consider this under the "convergence in probability," the theorem becomes:

Theorem. Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables $X_i : \Omega \to \mathbb{R}$ such that $X_n \xrightarrow{\mathbb{P}} X$, where $X : \Omega \to \mathbb{R}$ is a random variable, as $n \to \infty$. If $|X_n| \leq Y$, for random variable $Y : \Omega \to \mathbb{R}$, where $\mathbb{E}[|Y|] < \infty$, then:

$$\mathbb{E}[|X_n - X|] \to 0 \text{ as } n \to \infty,$$

and consequently:

$$\mathbb{E}[X_n] \to \mathbb{E}[X] \text{ as } n \to \infty.$$

Proof. Let $\epsilon > 0$ be arbitrary, we define:

$$\Omega_\epsilon := \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \leq \epsilon\},$$

and correspondingly:

$$\Omega_\epsilon^c := \{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\}.$$

By the definition of convergence in probability, there exists some $n \in \mathbb{N}^+$ such that $\mathbb{P}[|X_n - X| > \epsilon] < \epsilon$, so we have $\mathbb{P}(\Omega_\epsilon^c) < \epsilon$ with arbitrarily large n.
Also, since $\mathbb{E}[|Y|] < \infty$, we note that $|Y|$ must be bounded a.e., that is $|Y| < k$ for some $k \in \mathbb{R}^+$ a.e.
Then, we want to decompose our expectation as:

$$\begin{aligned}
\mathbb{E}[|X_n - X|] &= \int_\Omega |X_n(\omega) - X(\omega)| \, d\mathbb{P}(\omega) \\
&= \int_{\Omega_\epsilon} |X_n(\omega) - X(\omega)| \, d\mathbb{P}(\omega) + \int_{\Omega_\epsilon^c} |X_n(\omega) - X(\omega)| \, d\mathbb{P}(\omega) \\
&\leq \mathbb{P}(\Omega_\epsilon)\epsilon + \int_{\Omega_\epsilon^c} 2|Y(\omega)| \, d\mathbb{P}(\omega) \\
&\leq 1 \cdot \epsilon + 2k\epsilon \leq (2k+1)\epsilon.
\end{aligned}$$

Thus, as $n \to \infty$, $\mathbb{E}[|X_n - X|] < (2k+1)\epsilon$ for all $\epsilon > 0$, so $\mathbb{E}[|X_n - X|] \to 0$.
Afterwards, we shall note that:

$$|\mathbb{E}[X_n] - \mathbb{E}[X]| = |\mathbb{E}[X_n - X]| \leq \mathbb{E}[|X_n - X|] \to 0 \text{ as } n \to \infty,$$

so we have $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$. □

II Problem Set 2

Problem II.1. (Exercise 2.17 on [Øksendal]). If $X_t(\cdot):\Omega\to\mathbb{R}$ is a continuous stochastic process, then for $p>0$ the *p-th variation process* of X_t, $\langle X,X\rangle_t^{(p)}$ is defined by:

$$\langle X,X\rangle_t^{(p)}(\omega)=\lim_{\Delta t_k\to 0}\sum_{t_k\le t}\left|X_{t_{k+1}}(\omega)-X_{t_k}(\omega)\right|^p$$

as the limit in probability where $0=t_1<t_2<\cdots<t_n=n$ and $\Delta t_k=t_{k+1}-t_k$. In particular, if $p=1$, this process is called the *total variation process* and if $p=2$, it is called the *quadratic variation process*. For Brownian motion $B_t\in\mathbb{R}$, we now show that the quadratic variation process is simply:

$$\langle B,B\rangle_t(\omega)=\langle B,B\rangle_t^{(2)}(\omega)=t\text{ a.s.}$$

(a) Define:

$$\Delta B_k=B_{t_{k+1}}-B_{t_k},$$

and put:

$$Y(t,\omega)=\sum_{t_k\le t}\big(\Delta B_k(\omega)\big)^2.$$

Show that:

$$\mathbb{E}\left[\left(\sum_{t_k\le t}(\Delta B_k)^2-t\right)^2\right]=2\sum_{t_k\le t}(\Delta t_k)^2,$$

and deduce that $Y(t,\cdot)\to t$ in $L^2(P)$ as $\Delta t_k\to 0$.

Proof. Here, we first recall the property of Brownian motion so that:

$$\Delta B_k\sim\mathcal{N}(0,t_{k+1}-t_k)=\mathcal{N}(0,\Delta t_k).$$

Here, we note that the Brownian motions are independent, so we have:

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{t_k\le t}(\Delta B_k)^2-t\right)^2\right]&=\mathbb{E}\left[\sum_{t_k\le t}\big((\Delta B_k)^2-t\big)^2\right]=\sum_{t_k\le t}\mathbb{E}\big[\big((\Delta B_k)^2-t\big)^2\big]\\&=\sum_{t_k\le t}\mathbb{E}\big[(\Delta B_k)^4-2t(\Delta B_k)^2+t^2\big].\end{aligned}$$

Recall the fourth moment being $3\sigma^4=3(\Delta t_k)^2$, the second moment as $\sigma^2=\Delta t_k$, so we have the expectation as:

$$\mathbb{E}\left[\left(\sum_{t_k\le t}(\Delta B_k)^2-t\right)^2\right]=3(\Delta t_k)^2-2(\Delta t_k)^2+(\Delta t_k)^2=2(\Delta t_k)^2.$$

Hence, as we consider the expectation as integral, we have:

$$\int_\Omega\left(\sum_{t_k\le t}(\Delta B_k(\omega))^2-t\right)^2d\mathbb{P}(\omega)\to 0\text{ as }\Delta t_k\to 0,$$

so we have L^2 convergence that $Y(t,\cdot):=\sum_{t_k\le t}(\Delta B_k(\omega))^2\to t$, as required. ☐

(b) Use (a) to prove that a.a. paths of Brownian motion do not have a bounded variation on $[0, t]$, i.e. the total variation of Brownian motion is infinite, a.s.

Proof. First, we may obtain the inequality that:

$$\sum_{t_k \le t} |\Delta B_k(\omega)| = \sum_{t_k \le t} \frac{|\Delta B_k(\omega)|^2}{|\Delta B_k(\omega)|} \ge \frac{1}{\sup_{t_k \le t} |\Delta B_k(\omega)|} \sum_{t_k \le t} |\Delta B_k(\omega)|^2.$$

Again, note that we want $\Delta t_k \to 0$, then we have $|\Delta B_k(\omega)| \to 0$ for all $t_k \le t$, thus:

$$\begin{aligned}\langle B, B\rangle_t^{(1)}(\omega) = \lim_{\Delta t_k \to 0} \sum_{t_k \le t} |\Delta B_k(\omega)| &\ge \lim_{\Delta t_k \to 0} \frac{1}{\sup_{t_k \le t} |\Delta B_k(\omega)|} \sum_{t_k \le t} |\Delta B_k(\omega)|^2 \\ &= \langle B, B\rangle_t^{(2)}(\omega) \lim_{\Delta t_k \to 0} \frac{1}{\sup_{t_k \le t} |\Delta B_k(\omega)|} = t \lim_{\Delta t_k \to 0} \frac{1}{\sup_{t_k \le t} |\Delta B_k(\omega)|} = +\infty.\end{aligned}$$

Hence, we have the total variation of the Brownian motion being infinite almost surely. □

Problem II.2. (Exercise 2.18 on [Øksendal]).

(a) Let $\Omega = \{1, 2, 3, 4, 5\}$ and let $\mathcal{U}$ be the collection:

$$\mathcal{U} = \{\{1, 2, 3\}, \{3, 4, 5\}\}$$

of subsets of Ω. Find the smallest σ-algebra containing $\mathcal{U}$, *i.e.*, the σ-algebra $\mathcal{H}_{\mathcal{U}}$ generated by $\mathcal{U}$.

Solution. From the beginning, the σ-algebra must contain the empty set and its compliment, $\{\emptyset, \Omega\}$. Then, consider the sets in the collection and their (countable union), we have:

$$\{\emptyset, \{1, 2, 3\}, \{3, 4, 5\}, \{1, 2, 3, 4, 5\} = \Omega\}.$$

Then, consider the complimentary sets, we must have:

$$\{\emptyset, \{1, 2, 3\}, \{3, 4, 5\}, \Omega, \{4, 5\}, \{1, 2\}\},$$

while this would have created another union and a compliment, so we have:

$$\{\emptyset, \{3\}, \{1, 2\}, \{4, 5\}, \{1, 2, 3\}, \{3, 4, 5\}, \{1, 2, 4, 5\}, \Omega\}.$$

Now, one can verify that the above collection contains $\mathcal{U}$, has the empty set, compliments, and countable unions, so the σ-algebra is:

$$\mathcal{H}_{\mathcal{U}} = \boxed{\{\emptyset, \{3\}, \{1, 2\}, \{4, 5\}, \{1, 2, 3\}, \{3, 4, 5\}, \{1, 2, 4, 5\}, \Omega\}}.$$

⌟

(b) Define $X : \Omega \to \mathbb{R}$ by:

$$X(1) = X(2) = 0, \qquad X(3) = 10, \qquad X(4) = X(5) = 1.$$

Is X measurable with respect $\mathcal{H}_{\mathcal{U}}$?

Solution. $\boxed{\text{Yes}}$. By Problem I.1(a), since we have a (at most) countable image, we can check the preimage of each single value of output. Note that:

$$X^{-1}(0) = \{1, 2\} \in \mathcal{H}_{\mathcal{U}}, \qquad X^{-1}(10) = \{3\} \in \mathcal{H}_{\mathcal{U}}, \qquad \text{and } X^{-1}(1) = \{4, 5\} \in \mathcal{H}_{\mathcal{U}},$$

so X is $\mathcal{H}_{\mathcal{U}}$-measurable. ⌟

(c) Define $Y : \Omega \to \mathbb{R}$ by:

$$Y(1) = 0, \qquad Y(2) = Y(3) = Y(4) = Y(5) = 1.$$

Find the σ-algebra $\mathcal{H}_Y$ generated by Y.

Solution. Here, we may note that the preimage is discrete, so we consider the collection:

$$\mathcal{Y} = \{\{1\}, \{2, 3, 4, 5\}\},$$

and our solution is the σ-algebra generated by $\mathcal{Y}$, namely:

$$\mathcal{H}_{\mathcal{Y}} = \boxed{\{\emptyset, \{1\}, \{2, 3, 4, 5\}, \Omega\}}.$$

⌟

Problem II.3. Suppose $\{Z_k\}_{k=1}^{\infty}$ are independent $\mathcal{N}(0, 1)$ random variables. Show that $|Z_n(\omega)| = \mathcal{O}(\sqrt{\log(n)})$ as $n \to \infty$ almost surely.
Hint: You may need Borel-Cantelli lemma.

Proof. Here, we construct our set of events $\{A_k\}_{k=1}^{\infty}$. We let:

$$A_k := \{\omega \in \Omega : |Z_k| > 2\sqrt{\log k}\}.$$

Then, we note that:

$$\mathbb{P}(A_k) = 2\mathbb{P}(Z_k > 2\sqrt{\log k}) = 1 - \operatorname{erf}(\sqrt{2 \log k}),$$

and we want to show that $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < +\infty$.
Here, we first notice that $\mathbb{P}(A_2) \lessapprox 0.095891 \ll 0.25 = 1/2^2$, and we take their derivatives as:

$$\begin{aligned} \frac{d}{dk}\left[1 - \operatorname{erf}(\sqrt{2 \log k})\right] &= -\frac{2}{\sqrt{\pi}} \frac{d}{dk} \int_0^{\sqrt{2 \log k}} e^{-t^2} dt \\ &= -\frac{2}{\sqrt{\pi}} \exp(-2 \log k) \cdot \frac{1}{k\sqrt{2 \log k}} = -\frac{2/\sqrt{\pi}}{k^3 \sqrt{\log k}}. \end{aligned}$$

Note that when we take the derivative of $1/k^2$ with respect to k, we obtain $-2/k^3$, in which we have:

$$\frac{d}{dk}\left[\frac{1}{k^2}\right] = -\frac{2}{k^3} > -\frac{2/\sqrt{\pi}}{k^3\sqrt{\log k}} = \frac{d}{dk}\left[1 - \operatorname{erf}(\sqrt{2\log k})\right] \text{ for } k > 0.$$

Hence, we may conclude that:

$$\mathbb{P}(A_k) < \frac{1}{k^2} \text{ for all } k \geq 2.$$

Hence, we have:

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) \leq 1 + \sum_{k=2}^{\infty} \mathbb{P}(A_k) \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty,$$

by the convergence of harmonic series, so our sets A_k satisfies the condition Borel-Cantelli lemma. Now, since $\{Z_k\}_{k=1}^{\infty}$ is independent, we have:

$$\mathbb{P}\left(\limsup_{k\to\infty}(A_k)\right) = \mathbb{P}\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\right) = 0,$$

which means that:

$$\mathbb{P}(\{\omega \in \Omega : |Z_k(\omega)| > 2\sqrt{\log k}\}) \to 0 \text{ as } k \to \infty,$$

which implies that $|Z_k(\omega)| \leq 2\sqrt{\log k}$ for all $\omega \in \Omega \setminus N$ where N is a null set, and hence:

$$|Z_n(\omega)| \leq 2\sqrt{\log n} \text{ as } n \to \infty \text{ a.s.},$$

which completes the proof. □

Problem II.4. Let $\{B_t\}_{t\geq 0}$ be one-dimensional Brownian motion.

(a) Find the density of the random vector (B_s, B_t) where $0 < s < t < \infty$.

Solution. Here, for the density function, we are able to express the probability as:

$$\begin{aligned}
\mathbb{P}(B_s \in F_1, B_t \in F_2) &= \int_{F_1\times F_2} \rho(s,x)\rho(t-s,y-x)dxdy \\
&= \int_{F_1\times F_2} \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{|x|^2}{2s}\right) \cdot \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{|y-x|^2}{2(t-s)}\right) dxdy \\
&= \int_{F_1\times F_2} \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{|x|^2}{2s} - \frac{|y-x|^2}{2(t-s)}\right) dxdy.
\end{aligned}$$

Hence, the density function is:

$$\rho(s,t,x,y) = \boxed{\frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{|x|^2}{2s} - \frac{|y-x|^2}{2(t-s)}\right)}.$$

⌟

(b) Find the conditional density of the vector (B_s, B_t) where $0 < s < t < 1$ under the condition $B_1 = 0$.

Solution. Here, we consider the conditional probability as:

$$\mathbb{P}(B_s \in F_1, B_t \in F_2 \mid B_1 = 0) = \frac{\mathbb{P}(B_s \in F_1, B_t \in F_2, B_1 = 0)}{\mathbb{P}(B_1 = 0)}.$$

Hence, the density function will be given as:

$$\begin{aligned}\rho(s,t,x,y) &= \frac{\rho(s,x)\rho(t-s,y-x)\rho(1-t,0-y)}{\rho(1,0)} \\ &= \frac{\frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{|x|^2}{2s}\right)\cdot\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{|y-s|^2}{2(t-s)}\right)\cdot\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{|y|^2}{2(1-t)}\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{|0|^2}{2}\right)} \\ &= \boxed{\frac{1}{2\pi\sqrt{s(t-s)(1-t)}}\exp\left(-\frac{|x|^2}{2s}-\frac{|y-s|^2}{2(t-s)}-\frac{|y|^2}{2(1-t)}\right)}.\end{aligned}$$

⌟

(c) Consider the process $X_t = e^{-\alpha t/2}B_{e^{\alpha t}}$. Find the probability density of $(X_{t_1}, \cdots, X_{t_n})$.

Solution. Again, the vector of the Brownian motion is the random vector of a multi-normal distribution, that is:

$$(B_{e^{\alpha t_1}}, B_{e^{\alpha t_2}}, \cdots, B_{e^{\alpha t_n}}) \sim \mathcal{N}\big((0,0,\cdots,0),\mathbf{\Sigma}\big),$$

where $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$ is a positive definite variance matrix, now we consider the exponentials, so the distribution would be:

$$(X_{t_1}, \cdots, X_{t_n}) \sim \mathcal{N}\big((0,0,\cdots,0),\mathbf{\Sigma}\big),$$

hence, so the density function is:

$$\rho_{(X_{t_1},\cdots,X_{t_n})\sim\mathcal{N}\left((0,0,\cdots,0)\right)}(x_1,x_2,\cdots,x_n) = \boxed{2\pi|\mathbf{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(x_1,\cdots,x_n)^{\intercal}\mathbf{\Sigma}(x_1,\cdots,x_n)\right)}.$$

⌟

Problem II.5. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with mean 0 and variance σ^2. Denote $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$. Let $\{Z_n\}_{n\geq 1}$ be a square-integrable process *predictable* with respect to $\mathcal{F}_n$ (*i.e.*, Z_{n+1} is $\mathcal{F}_n$-measurable).

(a) Show that $Y_n = \sum_{k=1}^{n} Z_k X_k$ is a square integrable martingale.

Proof. First, we want to show that Y_n is square integrable, for each finite n, it is a finite sum of random variables, so we can reduce to the case of showing that $Z_k X_k$ is square integrable. Consider that:

$$\begin{aligned}\int_{\Omega} |Z_k(\omega)X_k(\omega)|^2 d\omega &= \int_{\{\omega\in\Omega:|\omega|\leq\delta\}} |Z_k(\omega)X_k(\omega)|^2 d\omega + \int_{\{\omega\in\Omega:|\omega|>\delta\}} |Z_k(\omega)X_k(\omega)|^2 d\omega \\ &\leq C_1 \int_{\{\omega\in\Omega:|\omega|\leq\delta\}} |Z_k(\omega)|^2 d\omega + C_2 \int_{\{\omega\in\Omega:|\omega|>\delta\}} |X_k(\omega)|^2 d\omega.\end{aligned}$$

Note that with choice of δ, Z_k will become bounded for larger then δ as it is square integrable, and X_k will be bounded for smaller than δ as it has mean of 0, hence the function is still square integrable.
For the martingale part, fir any $n \geq 1$ and $j > n$, we have the conditional expectation as:

$$\mathbb{E}[Y_j \mid Y_1, \cdots, Y_k] = \sum_{i=1}^{j} \mathbb{E}[Z_i X_i \mid Y_1, \cdots, Y_k] = \sum_{i=1}^{k} Z_i X_i + \sum_{i=k+1}^{j} \underbrace{Z_i \mathbb{E}[X_i]}_{=0} = Y_k,$$

hence we have shown that Y_n is martingale.
Therefore, $\{Y_n\}$ is a sequence of square integrable martingale. □

(b) Show that $\mathbb{E}[Y_n] = 0$ and that $\mathbb{E}[Y_n^2] = \sigma^2 \sum_{k=1}^{n} \mathbb{E}[Z_n^2]$.

Proof. Here, we may consider the expectation based on the different measure of X:

$$\mathbb{E}[Y_n] = \sum_{k=1}^{n} \mathbb{E}[Z_k X_k] = \sum_{k=1}^{n} \int_{\mathcal{F}_n} Z_k X_k \, d\mathbb{P} = \sum_{k=1}^{n} \left(\int_{\mathcal{F}_n} Z_k \, d\mathbb{P} \cdot \int_{\mathcal{F}_n} Z_k X_k \, d\mathbb{P} \right) = 0.$$

Then, we consider the second moment as (by independence):

$$\mathbb{E}[Y_n^2] = \mathrm{Var}[Y_n] = \sum_{k=1}^{n} \mathrm{Var}[Z_k] \, \mathrm{Var}[X_k] = \sum_{k=1}^{n} \sigma^2 \mathbb{E}[Z_k^2] = \sigma^2 \sum_{k=1}^{n} \mathbb{E}[Z_k^2],$$

which finishes the proof. □

(c) Let us assume $Z_k = \frac{1}{k}$. Is the martingale $\{Y_n\}_{n\geq 1}$ uniformly integrable?

Solution. Here, we may observe from (b) that we would have Y_n having expectation and variance as:

$$\mathbb{E}[Y_n] = 0 \text{ and } \mathbb{E}[Y_n^2] = \sigma^2 \sum_{k=1}^{n} \mathbb{E}[Z_n^2].$$

Hence, as $n \to \infty$, we have $\mathbb{E}[Y_n^2] < +\infty$ converging. Therefore, when we consider:

$$\lim_{m\to\infty} \sup_{i\geq 1} \left[\int_{|Y_i|\geq m} |Y_i| d\mathbb{P} \right],$$

where we have $\mathbb{P}(|Y_i| \geq m) \to 0$ as $m \to \infty$, and so the limit is zero and the martingale is uniformly integrable. ⌟

III Problem Set 3

Problem III.1. (Exercise 3.1 on [Øksendal]). Prove directly from the definition of Itô integrals that:

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

Hint: Note that:

$$\sum_j \Delta(s_j B_j) = \sum_j s_j \Delta B_j + \sum_j B_{j+1} \Delta s_j.$$

Proof. Here, from the definition, we note that s is already an elementary function, so we may consider the partition such that $\Delta t \to 0$:

$$\int_0^t s dB_s = \sum_j s_j \Delta B_j = \sum_j \Delta(s_j B_j) - \sum_j B_{j+1} \Delta s_j = tB_t - \int_0^t B_s ds,$$

as desired. □

Problem III.2. (Exercise 3.5 on [Øksendal]). Prove directly that:

$$M_t = B_t^2 - t$$

is an $\mathcal{F}_t$-martingale.

Proof. First, we want to show that the process is integrable, *i.e.*, for any fixed $t > 0$:

$$\mathbb{E}[|M_t|] = \mathbb{E}[|B_t^2 - t|] = \mathbb{E}[|\chi^2(t) - t|] < +\infty.$$

Then, we suppose any $s \leq t$ fixed, and recall that Brownian motions are martingale, let:

$$\begin{aligned}
\mathbb{E}[M_t \mid \mathcal{F}_s] &= \mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = \mathbb{E}[B_t^2 \mid \mathcal{F}_s] - t \\
&= \mathbb{E}[(B_t - B_s)^2 + 2B_t B_s - B_s^2 \mid \mathcal{F}_s] - t \\
&= \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + \mathbb{E}[2B_t B_s \mid \mathcal{F}_s] - \mathbb{E}[B_s^2 \mid \mathcal{F}_s] - t \\
&= (t - s) + 2B_s \mathbb{E}[B_t \mid \mathcal{F}_s] - B_s^2 - t = B_s^2 - s = M_s,
\end{aligned}$$

so M_t is an $\mathcal{F}_t$-martingale. □

Problem III.3. (Exercise 3.7 on [Øksendal]). A famous result of Itô (1951) gives the following formula for n times *iterated Itô integrals*:

$$n! \int \cdots \int_{0 \leq u_1 \leq \cdots \leq u_n \leq t} \left(\int \left(\int dB_{u_1} \right) dB_{u_2} \right) \cdots dB_{u_n} = t^{\frac{n}{2}} h_n \left(\frac{B_t}{\sqrt{t}} \right), \tag{2}$$

where h_n is the *Hermite polynomial* of degree n, defined by:

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right); \qquad n = 0, 1, 2, \cdots.$$

Thus $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$.

(a) Verify that in each of these n Itô integrals, the integrand satisfies the requirements for $\mathcal{V}$.

Proof. Here, we note that $h_n(x)$ is integrable, and we have:

$$f_n(t,\omega) = \frac{1}{(n-1)!} t^{\frac{n-1}{2}} h_{n-1}\left(\frac{B_t}{\sqrt{t}}\right),$$

we want to show:

- $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable.
  Note that for h_n is measurable over $\mathcal{B} \times \mathcal{F}$, so it is good.
- $f(t,\omega)$ is $\mathcal{F}_t$-adapted, *i.e.*, $\omega \mapsto f(t,\omega)$ is $\mathcal{F}_t$-measurable.
  Again, h_n is measurable of $\mathcal{F}$ with fixed ω, so it is good.
- $\mathbb{E}\left[\int_0^T f(t,\omega)^2 dt\right] < +\infty$.
  We have:
  $$\mathbb{E}\left[\int_0^T f(t,\omega)^2 dt\right] \leq nT^2 < +\infty.$$

Hence, the integrands satisfies the requirements of being $\mathcal{V}$. □

(b) Verify formula (2) for $n = 1, 2, 3$.

Proof.

- ($n = 1$:) We have:
  $$1! \int_0^t dB_{u_1} = B_t = \sqrt{t} \cdot \frac{B_t}{\sqrt{t}}.$$
- ($n = 2$:) We have:
  $$2! \int_0^t B_{u_2} dB_{u_2} = B_t^2 - t = t\left(\frac{B_t^2}{t} - 1\right)$$
- ($n = 3$:) We have:
  $$\begin{aligned} 3! \int_0^t \left(\frac{1}{2}B_{u_3}^2 - \frac{1}{2}u_3\right) dB_{u_3} &= 3\int_0^t B_{u_3}^2 dB_{u_3} - 3\int_0^t u_3 dB_{u_3} = B_t^3 - 3\int_0^t B_{u_3} du_3 + 3tB_t - 3\int_0^t B_{u_3} du_3 \\ &= B_t^3 - 3tB_t = t^{\frac{3}{2}}\left(\frac{B_t^3}{t^{\frac{3}{2}}} - 3\frac{B_t}{\sqrt{t}}\right). \end{aligned}$$

□

(c) Use (b) to prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

Proof. Note that Itô integrals are martingale, and since $B_t^3 - 3tB_t$ is an Itô integral, it is martingale. □

Problem III.4. Compute:

(a)
$$\mathbb{E}\left[B_s \int_0^t B_r dB_r\right].$$

Solution. Here, we have:

$$\mathbb{E}\left[B_s \int_0^t B_r dB_r\right] = \mathbb{E}\left[B_s \cdot \frac{1}{2}(B_t^2 - t)\right] = \frac{1}{2}\mathbb{E}[B_sB_t^2 - tB_s] = \frac{1}{2}\mathbb{E}[B_sB_t^2] - \frac{1}{2}t\mathbb{E}[B_s] = \frac{1}{2}\mathbb{E}[B_sB_t^2].$$

Now, we consider two distinctive cases for $\mathbb{E}[B_sB_t^2]$:

- ($s \le t$:) We have:

$$\begin{aligned}\mathbb{E}[B_sB_t^2] &= \mathbb{E}[B_s(B_t - B_s)^2 - B_s^3 + 2B_s^2B_t] = \mathbb{E}[B_s]\mathbb{E}[(B_t - B_s)^2] - \mathbb{E}[B_s^3] + 2\mathbb{E}[B_s^2B_t] \\ &= 0 \cdot (t-s) - 0 + 2\mathbb{E}[B_s^2B_t] = 2\mathbb{E}[B_s^2B_t] \\ &= 2\mathbb{E}[B_s^2(B_t - B_s) + B_s^3] = 2\mathbb{E}[B_s^2]\mathbb{E}[B_t - B_s] + 2\mathbb{E}[B_s^3] = 2 \cdot s \cdot 0 + 0 = 0.\end{aligned}$$

- ($s > t$:) Otherwise, we have:

$$\mathbb{E}[B_sB_t^2] = \mathbb{E}[B_t^2(B_s - B_t) + B_t^3] = \mathbb{E}[B_t^2]\mathbb{E}[B_s - B_t] + \mathbb{E}[B_t^3] = t \cdot 0 + 0 = 0.$$

Hence, we have the expectation evaluated as $\boxed{0}$. ⌟

(b)
$$\mathbb{E}\left[\left(B_s \int_0^t B_r dB_r\right)^2\right] \text{ where } s \le t.$$

Solution. Here, we have:

$$\begin{aligned}\mathbb{E}\left[\left(B_s \int_0^t B_r dB_r\right)^2\right] &= \mathbb{E}\left[\left(B_s \cdot \frac{1}{2}(B_t^2 - t)\right)^2\right] = \frac{1}{4}\mathbb{E}[B_s^2(B_t^4 - 2t^2B_t^2 + t^2)] \\ &= \frac{1}{4}\mathbb{E}[B_s^2B_t^4 - 2t^2B_s^2B_t^2 + t^2B_s^2] = \frac{1}{4}\mathbb{E}[B_s^2B_t^4] - \frac{1}{2}t^2\mathbb{E}[B_s^2B_t^2] + \frac{1}{4}t^2\mathbb{E}[B_s^2] \\ &= \frac{1}{4}\mathbb{E}[B_s^2B_t^4] - \frac{1}{2}t^2\mathbb{E}[B_s^2B_t^2] + \frac{1}{4}t^2 s.\end{aligned}$$

Now, we investigate the two respective expectations.

- For $\mathbb{E}[B_s^2 B_t^2]$, we have:

$$\begin{aligned}\mathbb{E}[B_s^2 B_t^2] &= \mathbb{E}[B_s^2(B_t - B_s)^2 - B_s^4 + 2B_s^3 B_t] = \mathbb{E}[B_s^2]\mathbb{E}[(B_t - B_s)^2] - \mathbb{E}[B_s^4] + 2\mathbb{E}[B_s^3 B_t] \\ &= s(t-s) - 3s^2 + 2\mathbb{E}[B_s^3(B_t - B_s) + B_s^4] = s(t-s) - 3s^2 + 2\mathbb{E}[B_s^3(B_t - B_s)] + 2\mathbb{E}[B_s^4] \\ &= s(t-s) - 3s^2 + 2\cdot 0 \cdot (t-s) + 2\cdot 3s^2 = st + 2s^2.\end{aligned}$$

- For $\mathbb{E}[B_s^2 B_t^4]$, we have:

$$\begin{aligned}\mathbb{E}[B_s^2 B_t^4] &= \mathbb{E}[B_s^2(B_t - B_s)^4 + 4B_t^3 B_s^3 - 6B_t^2 B_s^4 + 4B_t B_s^5 - B_s^6] \\ &= \mathbb{E}[B_s^2(B_t - B_s)^4] + 4\mathbb{E}[B_t^3 B_s^3] - 6\mathbb{E}[B_t^2 B_s^4] + 4\mathbb{E}[B_t B_s^5] - \mathbb{E}[B_s^6] \\ &= s\cdot 3\cdot (t-s)^2 + 4\mathbb{E}[B_t^3 B_s^3] - 6\mathbb{E}[B_t^2 B_s^4] + 4\mathbb{E}[B_t B_s^5] - 15s^3 \\ &= 3t^2 s - 6ts^2 - 12s^3 + 4\mathbb{E}[B_t^3 B_s^3] - 6\mathbb{E}[B_t^2 B_s^4] + 4\mathbb{E}[B_t B_s^5].\end{aligned}$$

Now, we have to evaluate the next terms:

  – For $\mathbb{E}[B_t B_s^5]$, we have:

$$\mathbb{E}[B_t B_s^5] = \mathbb{E}[B_s^5(B_t - B_s) + B_s^6] = 15s^3.$$

  – For $\mathbb{E}[B_t^2 B_s^4]$, we have:

$$\mathbb{E}[B_t^2 B_s^4] = \mathbb{E}[B_s^4(B_t - B_s)^2 + 2B_s^5 B_t - B_s^6] = 3s^2\cdot (t-s) + 30s^3 - 15s^3 = 3ts^2 + 12s^3.$$

  – For $\mathbb{E}[B_t^3 B_s^3]$, we have:

$$\begin{aligned}\mathbb{E}[B_t^3 B_s^3] &= \mathbb{E}[B_s^3(B_t - B_s)^3 + 3B_s^4 B_t^2 - 3B_s^5 B_t + B_s^6] \\ &= 0 + 3(3ts^2 + 12s^3) - 3(15s^3) + 15s^3 = 9ts^2 + 6s^3.\end{aligned}$$

Now, we can combine all the calculations together:

$$\begin{aligned}\mathbb{E}[B_s^2 B_t^4] &= 3t^2 s - 6ts^2 - 12s^3 + 4(9ts^2 + 6s^3) - 6(3ts^2 + 12s^3) + 4(15s^3) \\ &= 3t^2 s + 12ts^2.\end{aligned}$$

Hence, we may conclude that:

$$\mathbb{E}\left[\left(B_s \int_0^t B_r dB_r\right)^2\right] = \frac{1}{4}(3t^2 s + 12ts^2) - \frac{1}{2}t^2(st + 2s^2) + \frac{1}{4}t^2 s = \boxed{t^2 s + 3ts^2 - \frac{1}{2}st^3 + s^2t^2}.$$

⌟

Problem III.5. (Exercise 3.17 on [Øksendal]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable with $\mathbb{E}[|X|] < \infty$. If $\mathcal{G} \subset \mathcal{F}$ is a *finite* σ-algebra, then there exists a partition $\Omega = \bigcup_{i=1}^n G_n$ such that $\mathcal{G}$ consists of $\emptyset$ and unions of some (or all) of $G_1, \cdots, G_n$.

(a) Explain why $\mathbb{E}[X \mid \mathcal{G}](\omega)$ is constant on each G_i.

Proof. Here, we may consider G as a random variable, namely:

$$G = \sum_{i=1}^{n} a_i \mathbb{1}_{G_i}, \text{ where } G_i \in \mathcal{G}.$$

Then, the conditional expectation for each given $\omega \in G_j$ is:

$$\mathbb{E}[X|\mathcal{G}](\omega) = \sum_{i=1}^{n} a_i \mathbb{1}_{G_i} = a_j.$$

□

(b) Assume that $\mathbb{P}[G_i] > 0$. Show that:

$$\mathbb{E}[X \mid \mathcal{G}](\omega) = \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \text{ for } \omega \in G_i.$$

Proof. Here, we just need to verify that:

$$\int_{G_i} \mathbb{E}[X \mid \mathcal{G}](\omega) d\mathbb{P}(\omega) = \int_{G_i} \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} d\mathbb{P} = \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \int_{G_i} d\mathbb{P} = \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \cdot \mathbb{P}(G_i) = \int_{G_i} X d\mathbb{P},$$

so it satisfies the condition for conditional expectation. □

(c) Suppose X assumes only finitely many values $a_1, \cdots, a_m$. Then from elementary probability theory:

$$\mathbb{E}[X \mid G_i] = \sum_{k=1}^{m} a_k \mathbb{P}[X = a_k \mid G_i].$$

Compare with (b) and verify that:

$$\mathbb{E}[X \mid G_i] = \mathbb{E}[X \mid \mathcal{G}](\omega) \text{ for } \omega \in G_i.$$

Thus, we may regard the conditional expectation as defined as a (substantial) generalization of the conditional expectation in the elementary probability theory.

Proof. Here, consider $\omega \in G_i$ being arbitrary, we have:

$$\begin{aligned} \mathbb{E}[X \mid \mathcal{G}](\omega) = \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} &= \frac{\sum_{k=1}^{m} a_k \mathbb{P}(X = a_k \wedge a_k \in G_i)}{\mathbb{P}(G_i)} \\ &= \sum_{k=1}^{m} \frac{a_k \mathbb{P}(X = a_k \wedge a_k \in G_i)}{\mathbb{P}(G_i)} = \sum_{k=1}^{m} a_k \mathbb{P}(X = a_k \mid G_i) = \mathbb{E}[X \mid G_i], \end{aligned}$$

so the general definition is aligned to the elementary probability theory definition. □

Problem III.6. (Exercise 3.18 on [Øksendal]). Let B_t be 1-dimensional Brownian motion and let $\sigma \in \mathbb{R}$ be constant. Prove directly from the definition that:

$$M_t := \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right); \qquad t \geq 0$$

is a martingale.
Hint: If $s > t$, then $\mathbb{E}[\exp(\sigma B_s - \frac{1}{2}\sigma^2 s) \mid \mathcal{F}_t] = \mathbb{E}\big[\exp\big(\sigma(B_s - B_t)\big) \times \exp(\sigma B_t - \frac{1}{2}\sigma^2 s) \mid \mathcal{F}_t\big]$.

Proof. Here, by the hint, we may notice that:

$$\begin{aligned}
\mathbb{E}[M_s \mid \mathcal{F}_t] &= \mathbb{E}\left[\exp\left(\sigma B_s - \frac{1}{2}\sigma^2 s\right) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(\sigma(B_s - B_t)\right) \cdot \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 s\right) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\big[\exp\left(\sigma(B_s - B_t)\right) \mid \mathcal{F}_t\big] \cdot \mathbb{E}\left[\exp\left(\sigma B_t - \frac{1}{2}\sigma^2 s\right) \mid \mathcal{F}_t\right] \\
&= \exp\left(\frac{1}{2}\sigma^2(s-t)\right) \cdot \exp\left(-\frac{1}{2}\sigma^2 s\right) \cdot \mathbb{E}[\exp(\sigma B_t) \mid \mathcal{F}_t] \\
&= \exp\left(-\frac{1}{2}\sigma^2 t\right) \cdot \exp(\sigma B_t) \\
&= \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right) = M_t.
\end{aligned}$$

Moreover, we consider the expectation of M_t, namely:

$$\begin{aligned}
\mathbb{E}[|M_t|] &= \mathbb{E}\left[\left|\exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)\right|\right] = \mathbb{E}\left[\exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)\right] \\
&= \exp\left(-\frac{1}{2}\sigma^2 t\right)\mathbb{E}[\exp(\sigma B_t)] = \exp\left(-\frac{1}{2}\sigma^2 t\right)\exp\left(\frac{1}{2}\sigma^2 t\right) = 1 < +\infty.
\end{aligned}$$

Hence, we have shown that M_t is martingale. ☐

IV Problem Set 4

Problem IV.1. (Exercise 4.1 on [Øksendal]). Use Itô's formula to write the following stochastic processes Y_t in the standard form:

$$dY_t = u(t,\omega)dt + v(t,\omega)dB_t$$

for suitable choices of $u \in \mathbb{R}^n$, $v \in \mathbb{R}^{n\times m}$ and dimensions n, m:

(a) $Y_t = B_t^2$, where B_t is 1-dimensional.

Solution. Here, we note that:

$$Y_t = B_t^2 = \int_0^t ds + 2\int_0^t B_s dB_s,$$

hence it is in standard form as:

$$dY_t = \boxed{dt + B_t dt}.$$

⌟

(b) $Y_t = 2 + t + e^{B_t}$, where B_t is 1-dimensional.

Solution. Here, we may apply **Itô formula**, namely:

$$\begin{aligned} dY_t &= \frac{\partial}{\partial t}[2+t+e^{B_t}]dt + \frac{\partial}{\partial x}[2+t+e^{B_t}]dB_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}[2+t+e^{B_t}](dB_t)^2 \\ &= dt + e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt = \boxed{\left(1+\frac{1}{2}e^{B_t}\right)dt + e^{B_t}dB_t}. \end{aligned}$$

⌟

(c) $Y_t = B_1^2(t) + B_2^2(t)$, where (B_1, B_2) is 2-dimensional.

Solution. Here, we may apply the **general Itô formula** as:

$$\begin{aligned} dY_t = &\frac{\partial}{\partial t}[B_1^2(t)+B_2^2(t)]dt + \frac{\partial}{\partial B_1}[B_1^2(t)+B_2^2(t)]dB_1 + \frac{\partial}{\partial B_2}[B_1^2(t)+B_2^2(t)]dB_2 + \\ &\frac{1}{2}\frac{\partial^2}{\partial B_1^2}[B_1^2(t)+B_2^2(t)](dB_1)^2 + \frac{1}{2}\frac{\partial^2}{\partial B_2^2}[B_1^2(t)+B_2^2(t)](dB_1)^2 + \frac{\partial^2}{\partial B_1 \partial B_2}[B_1^2(t)+B_2^2(t)](dB_1 dB_2) \\ = &\, 0dt + 2B_1 dB_1 + 2B_2 dB_2 + dt + dt + 0\delta_{1,2}dt \\ = &\boxed{2dt + 2B_1(t)dB_1(t) + 2B_2(t)dB_2(t)}. \end{aligned}$$

⌟

(d) $Y_t = (t_0 + t, B_t)$, where B_t is 1-dimensional.

Solution. Here, we need to consider the process component-wise, denoted $Y_t = (Y_t^{(1)}, Y_t^{(2)})$.

For $Y_t^{(1)}$, we have:

$$d(Y_t^{(1)}) = \frac{\partial}{\partial t}[t_0 + t]dt + \frac{\partial}{\partial B_t}[t_0 + t]dB_t + \frac{1}{2}\frac{\partial}{\partial B_t^2}[t_0 + t](dB_t)^2 = dt.$$

For $Y_t^{(2)}$, we have:

$$d(Y_t^{(2)}) = \frac{\partial}{\partial t}[B_t]dt + \frac{\partial}{\partial B_t}[B_t]dB_t + \frac{1}{2}\frac{\partial}{\partial B_t^2}[B_t](dB_t)^2 = dB_t.$$

Hence, the process can be written in standard form as:

$$dY_t = \boxed{\begin{pmatrix}1\\0\end{pmatrix} dt + \begin{pmatrix}0\\1\end{pmatrix} dB_t}.$$

(e) $Y_t = \big(B_1(t) + B_2(t) + B_3(t), B_2^2(t) - B_1(t)B_3(t)\big)$, where (B_1, B_2, B_3) is 3-dimensional.

Solution. Again, we shall consider the process component-wise, denoted $Y_t = (Y_t^{(1)}, Y_t^{(2)})$.
For $Y_t^{(1)}$, we have:

$$\begin{aligned} dY_t^{(1)} = &\frac{\partial}{\partial t}[B_1 + B_2 + B_3]dt + \frac{\partial}{\partial B_1}[B_1 + B_2 + B_3]dB_1 + \frac{\partial}{\partial B_2}[B_1 + B_2 + B_3]dB_2 + \frac{\partial}{\partial B_3}[B_1 + B_2 + B_3]dB_3 + \\ &\frac{1}{2}\frac{\partial}{\partial B_1^2}[B_1 + B_2 + B_3](dB_1)^2 + \frac{1}{2}\frac{\partial}{\partial B_2^2}[B_1 + B_2 + B_3](dB_2)^2 + \frac{1}{2}\frac{\partial}{\partial B_3^2}[B_1 + B_2 + B_3](dB_3)^2 + \\ &\frac{\partial}{\partial B_1 \partial B_2}[B_1 + B_2 + B_3]dB_1 dB_2 + \frac{\partial}{\partial B_1 \partial B_3}[B_1 + B_2 + B_3]dB_1 dB_3 + \frac{\partial}{\partial B_2 \partial B_3}[B_1 + B_2 + B_3]dB_2 dB_3 \\ = &dB_1(t) + dB_2(t) + dB_3(t). \end{aligned}$$

For $Y_t^{(2)}$, we have:

$$\begin{aligned} dY_t^{(2)} = &\frac{\partial}{\partial t}[B_2^2 - B_1 B_3]dt + \frac{\partial}{\partial B_1}[B_2^2 - B_1 B_3]dB_1 + \frac{\partial}{\partial B_2}[B_2^2 - B_1 B_3]dB_2 + \frac{\partial}{\partial B_3}[B_2^2 - B_1 B_3]dB_3 + \\ &\frac{1}{2}\frac{\partial}{\partial B_1^2}[B_2^2 - B_1 B_3](dB_1)^2 + \frac{1}{2}\frac{\partial}{\partial B_2^2}[B_2^2 - B_1 B_3](dB_2)^2 + \frac{1}{2}\frac{\partial}{\partial B_3^2}[B_2^2 - B_1 B_3](dB_3)^2 + \\ &\frac{\partial}{\partial B_1 \partial B_2}[B_2^2 - B_1 B_3]dB_1 dB_2 + \frac{\partial}{\partial B_1 \partial B_3}[B_2^2 - B_1 B_3]dB_1 dB_3 + \frac{\partial}{\partial B_2 \partial B_3}[B_2^2 - B_1 B_3]dB_2 dB_3 \\ = &- B_3 dB_1 + 2B_2 dB_2 - B_1 dB_3 + (dB_2)^2 = dt - B_3(t)dB_1(t) + 2B_2(t)dB_2(t) - B_1(t)dB_3(t). \end{aligned}$$

Hence, when we combine the process together, we have:

$$dY_t = \boxed{\begin{pmatrix}0\\1\end{pmatrix} dt + \begin{pmatrix}1\\-B_3(t)\end{pmatrix} dB_1(t) + \begin{pmatrix}1\\2B_2(t)\end{pmatrix} dB_2(t) + \begin{pmatrix}1\\-B_1(t)\end{pmatrix} dB_3(t)}.$$

Problem IV.2. (Exercise 4.2 on [Øksendal]). Use Itô formula to prove that:

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

Proof. Here, we write B_t^3 in terms of differential form:

$$dB_t^3 = \frac{\partial}{\partial t}[B_t^3]dt + \frac{\partial}{\partial B_t}[B_t^3]dB_t + \frac{1}{2}\cdot\frac{\partial^2}{\partial B_t^2}[B_t^3](dB_t)^2 = 3B_t dt + 3B_t^2 dB_t,$$

and hence if we were to write them in terms of standard form, we have:

$$B_t^3 = 3\int_0^t B_s ds + 3\int_0^t B_s^2 dB_s,$$

and if we were to divide everything by 3 and move around, we have:

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds,$$

as desired. □

Problem IV.3. (Exercise 4.3 on [Øksendal]). Let X_t, Y_t be Itô processes in $\mathbb{R}$. Prove that:

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t.$$

Deduce the following general *integration by parts formula*:

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s.$$

Proof. Here, we may use the **general Itô formula** to find the differential form as:

$$\begin{aligned} d(X_tY_t) &= \frac{\partial}{\partial t}[X_tY_t]dt + \frac{\partial}{\partial X_t}[X_tY_t]dX_t + \frac{\partial}{\partial Y_t}[X_tY_t]dY_t + \\ &\quad \frac{1}{2}\frac{\partial^2}{\partial X_t^2}[X_tY_t](dX_t)^2 + \frac{1}{2}\frac{\partial^2}{\partial Y_t^2}[X_tY_t](dY_t)^2 + \frac{\partial^2}{\partial X_t \partial Y_t}[X_tY_t]dX_t dY_t \\ &= Y_t dX_t + X_t dY_t + dX_t dY_t. \end{aligned}$$

Then, we can write the differential form in standard form:

$$X_tY_t = X_0Y_0 + \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t dX_s \cdot dY_s.$$

Then, we can move around the terms to get the *integration by parts formula*:

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s.$$ □

Problem IV.4. (Exercise 4.4 on [Øksendal]). Exponential martingales.
Suppose $\theta(t,\omega) = \left(\theta_1(t,\omega), \cdots, \theta_n(t,\omega)\right) \in \mathbb{R}^n$ with $\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for $k = 1, \cdots, n$, where $T \leq \infty$. Define:

$$Z_t = \exp\left[\int_0^t \theta(s,\omega)dB(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega)ds\right]; \qquad 0 \leq t \leq T,$$

where $B(s) \in \mathbb{R}^n$ and $\theta^2 = \theta \cdot \theta$ as the dot product.

(a) Use Itô's formula to prove that:

$$dZ_t = Z_t\theta(t,\omega)dB(t).$$

Proof. Here, we first consider another process X_t such that:

$$dX_t = \theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt.$$

Here, we have $Z_t = \exp(X_t)$, and we use the Itô formula on a given process:

$$\begin{aligned}
dZ_t &= \frac{\partial}{\partial t}[\exp(X_t)]dt + \frac{\partial}{\partial X_t}[\exp(X_t)]dX_t + \frac{1}{2}\frac{\partial}{\partial X_t^2}[\exp(X_t)](dX_t)^2 \\
&= 0dt + \exp(X_t)dX_t + \frac{1}{2}\exp(X_t)(dX_t)^2 \\
&= \exp(X_t)\left(\theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt\right) + \frac{1}{2}\exp(X_t)\left(\theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt\right)^2 \\
&= \exp(X_t)\theta(t,\omega)dB(t) - \frac{1}{2}\exp(X_t)\theta^2(t,\omega)dt + \frac{1}{2}\exp(X_t)\theta^2(t,\omega)(dB(t))^2 - \\
&\quad \frac{1}{4}\exp(X_t)\theta^3(t,\omega)dB(t)dt + \frac{1}{8}\exp(X_t)\theta^4(t,\omega)(dt)^2 \\
&= \exp(X_t)\theta(t,\omega)dB(t) - \frac{1}{2}\exp(X_t)\theta^2(t,\omega)dt + \frac{1}{2}\exp(X_t)\theta^2(t,\omega)dt \\
&= \exp(X_t)\theta(t,\omega)dB(t) = Z_t\theta(t,\omega)dB(t),
\end{aligned}$$

as desired. □

(b) Deduce that Z_t is a martingale for $t \leq T$, provided that:

$$Z_t\theta_k(t,\omega) \in \mathcal{V}[0,T] \text{ for } 1 \leq k \leq n.$$

Proof. By part (a), we note that Z_t can be written as:

$$Z_t\theta_k(t,\omega) = \int_0^t Z_s\theta(t,\omega)dB(t) = \int_0^t \sum_{k=1}^n Z_s\theta_k(t,\omega)dB_k(t) = \sum_{k=1}^n \int_0^t Z_s\theta_k(t,\omega)dB_k(t).$$

Note that since $Z_s\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for all k, the integral $\int_0^t Z_s\theta_k(t,\omega)dB_k(t)$ must be martingale, and a finite sum of martingale is still martingale. □

V Problem Set 5

Problem V.1. (Exercise 4.13 on [Øksendal]). Let $dX_t = u(t,\omega)dt + dB_t$, where $u \in \mathbb{R}$ and $B_t \in \mathbb{R}$, be an Itô process and assume for simplicity that u is bounded. Then we know that unless $u = 0$ the process X_t is not an $\mathcal{F}_t$-martingale. However, it turns out that we can construct an $\mathcal{F}_t$-martingale from X_t by multiplying by a suitable exponential martingale. More precisely, define:

$$Y_t = X_t M_t,$$

where:

$$M_t = \exp\left(-\int_0^t u(r,\omega)dB_r - \frac{1}{2}\int_0^t u^2(r,\omega)dr\right).$$

Use Itô's formula to prove that Y_t is an $\mathcal{F}_t$-martingale.

Proof. Here, we think about the Itô formula on Y_t by considering the product rule:

$$dY_t = d(X_t M_t) = X_t dM_t + M_t dX_t + dX_t dM_t$$

Recall from Problem IV.4(a), we have:

$$dM_t = -M_t u(t,\omega)dB_t,$$

and hence we can continue the product rule as:

$$\begin{aligned} dY_t &= X_t M_t\big(-u(t,\omega)dB_t\big) + M_t\big(u(t,\omega)dt + dB_t\big) + \big(u(t,\omega)dt + dB_t\big)M_t\big(-u(t,\omega)dB_t\big) \\ &= -X_t M_t u(t,\omega)dB_t + M_t u(t,\omega)dt + M_t dB_t - M_t u(t,\omega)dt \\ &= M_t\big(1 - X_t u(t,\omega)\big)dB_t. \end{aligned}$$

Hence, the Itô formula of Y_t contains to dt terms, and recall from Problem IV.4(b), since u is a Itô process, so M_t is martingale, thus $\mathbb{E}[|M_t|] < +\infty$. Consider for X_t that:

$$\begin{aligned} \mathbb{E}[|X_t|] &= \mathbb{E}\left[\left|\int_0^t u(r,\omega)dr + \int_0^t dB_r\right|\right] \\ &\leq \mathbb{E}\left[\left|\int_0^t u(r,\omega)dr\right|\right] + \mathbb{E}\left[\left|\int_0^t dB_r\right|\right] \leq \mathbb{E}\left[\int_0^t \big|u(r,\omega)\big|dr\right] + \mathbb{E}[|B_t|] < +\infty, \end{aligned}$$

since $u(r,\omega)$ is bounded and $\mathbb{E}[|B_t|^2] = t$, so we have $\mathbb{E}[|X_t M_t|] \leq \mathbb{E}[|X_t|] \cdot \mathbb{E}[|M_t|] < +\infty$, hence have proven that Y_t is, in fact, a $\mathcal{F}_t$ martingale. □

Problem V.2. (Exercise 4.16 on [Øksendal]). If Y is an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|Y|^2] < \infty$, then the process:

$$M_t := \mathbb{E}[Y \mid \mathcal{F}_t]; \qquad 0 \leq t \leq T$$

is a martingale with respect to $\{\mathcal{F}_t\}_{0\leq t\leq T}$.

(a) Show that $\mathbb{E}[M_t^2] < \infty$ for all $t \in [0, T]$.

Proof. Note we have $\mathcal{F}_t$ as a σ-algebra, so we have:

$$\mathbb{E}\left[(\mathbb{E}[Y \mid \mathcal{F}_t])^2\right] \leq \mathbb{E}[Y^2] < +\infty,$$

as desired. □

(b) According to the martingale representation theorem, there exists a unique process $g(t, \omega) \in \mathcal{V}(0, T)$ such that:

$$M_t = \mathbb{E}[M_0] + \int_0^t g(s, \omega) dB(s); \qquad t \in [0, T].$$

Find g in the following cases:

1. $Y(\omega) = B^2(T)$.
2. $Y(\omega) = B^3(T)$.
3. $Y(\omega) = \exp\left(\sigma B(T)\right)$, where $\sigma \in \mathbb{R}$ is a constant.
   Hint: Use that $\exp\left(\sigma B(t) - \frac{1}{2}\sigma^2 t\right)$ is a martingale.

Solution.

1. Now, we have:

   $$M_t = \mathbb{E}[B_T^2 \mid \mathcal{F}_t].$$

   Here, we decompose that:

   $$B_T^2 = \big(B_t + (B_T - B_t)\big)^2 = B_t^2 + 2B_t(B_T - B_t) + (B_T - B_t)^2,$$

   so we have the conditional expectation as:

   $$\begin{aligned}\mathbb{E}[B_T^2 \mid \mathcal{F}_t] &= \mathbb{E}[B_t^2 + 2B_t(B_T - B_t) + (B_T - B_t)^2 \mid \mathcal{F}_t] \\ &= \mathbb{E}[B_t^2 \mid \mathcal{F}_t] + 2\mathbb{E}[B_t \mid \mathcal{F}_t]\mathbb{E}[B_T - B_t \mid \mathcal{F}_t] + \mathbb{E}[(B_T - B_t)^2 \mid \mathcal{F}_t] \\ &= B_t^2 + 2B_t\mathbb{E}[B_t - B_t] + \mathbb{E}[(B_T - B_t)^2] = B_t^2 + T - t.\end{aligned}$$

   Then, we apply the Itô formula and obtain that:

   $$dM_t = -dt + 2B_t dB_t + \frac{1}{2} \cdot 2dt = 2B_t dB_t,$$

   hence we have $g(s, \omega) = \boxed{2B_s(\omega)}$.

2. Now, we have:

   $$M_t = \mathbb{E}[B_T^3 \mid \mathcal{F}_t],$$

   and we similarly construct the decomposition as:

   $$B_T^3 = \big(B_t + (B_T - B_t)\big)^3 = B_t^3 + 3B_t^2(B_T - B_t) + 3B_t(B_T - B_t)^2 + (B_T^2 - B_t)^3.$$

Now, we apply the conditional expectation as:

$$\begin{aligned}\mathbb{E}[B_T^3 \mid \mathcal{F}_t] &= \mathbb{E}[B_t^3 + 3B_t^2(B_T - B_t) + 3B_t(B_T - B_t)^2 + (B_T^2 - B_t)^3 \mid \mathcal{F}_t] \\ &= \mathbb{E}[B_t^3 \mid \mathcal{F}_t] + 3\mathbb{E}[B_t^2 \mid \mathcal{F}_t]\mathbb{E}[B_T - B_t \mid \mathcal{F}_t] \\ &\quad + 3\mathbb{E}[B_t \mid \mathcal{F}_t]\mathbb{E}[(B_T - B_t)^2 \mid \mathcal{F}_t] + \mathbb{E}[(B_T^2 - B_t)^3 \mid \mathcal{F}_t] \\ &= B_t^3 + 3B_t(T-t) + 3B_t^2 \cdot 0 + T - t = B_t^3 + 3TB_t - 3tB_t.\end{aligned}$$

Then, we apply the Itô formula and obtain that:

$$\begin{aligned}dM_t &= -3B_t dt + (3B_t^2 + 3T - 3t)dB_t + \frac{1}{2} \cdot 6B_t dt \\ &= 3(B_t^2 + T - t)dB_t,\end{aligned}$$

and hence we have $g(s, \omega) = \boxed{3(B_t^2 + T - t)}$.

3. Here, we have:

$$M_t = \mathbb{E}[\exp(\sigma B_T) \mid \mathcal{F}_t],$$

and we consider that:

$$\exp(\sigma B_T) = \exp\big(\sigma(B_t + (B_T - B_t))\big) = \exp(\sigma B_t)\exp\big(\sigma(B_T - B_t)\big),$$

and we hence have that:

$$\begin{aligned}\mathbb{E}[\exp(\sigma B_T) \mid \mathcal{F}_t] &= \mathbb{E}[\exp(\sigma B_t)\exp\big(\sigma(B_T - B_t)\big) \mid \mathcal{F}_t] \\ &= \mathbb{E}[\exp(\sigma B_t) \mid \mathcal{F}_t] \cdot \mathbb{E}[\exp\big(\sigma(B_T - B_t)\big) \mid \mathcal{F}_t] \\ &= \exp(\sigma B_t) \cdot \exp\left(\frac{\sigma^2(T-t)}{2}\right).\end{aligned}$$

Hence, we apply Itô formula to obtain that:

$$dM_t = M_t\left(-\frac{\sigma^2}{2}\right)dt + M_t \cdot \sigma dB_t + \frac{1}{2}M_t \cdot \sigma^2 dt = M_t \cdot \sigma dB_t,$$

and hence we have $g(s, \omega) = \boxed{\sigma \exp(\sigma B_t) \cdot \exp\left(\frac{\sigma^2(T-t)}{2}\right)}$.

⌟

Problem V.3. (Exercise 5.7 on [Øksendal]). The *mean-reverting Ornstein-Uhlenbeck process* is the solution X_t of the stochastic differential equation:

$$dX_t = (m - X_t)dt + \sigma dB_t,$$

where m, σ are real constants, and $B_t \in \mathbb{R}$.

(a) Solve this equation using the integrating factor similar to e^t.

Solution. Here, we multiply by the integration factor that:

$$F_t = \exp(t), \qquad \text{and so } dF_t = \exp(t)dt.$$

Then, we consider the product rule as:

$$\begin{aligned} d(F_t X_t) &= F_t dX_t + X_t dF_t + dF_t dX_t \\ &= \exp(t)\big((m - X_t)dt + \sigma dB_t\big) + \exp(t)X_t dt + \exp(t)dt\big((m - X_t)dt + \sigma dB_t\big) \\ &= \exp(t)m dt + \exp(t)\sigma dB_t. \end{aligned}$$

Thereby, we write the equation in standard form:

$$\begin{aligned} F_t X_t &= F_0 X_0 + m\int_0^t \exp(s)ds + \sigma\int_0^t \exp(x)dB_s = F_0X_0 + m\big(\exp(t) - 1\big) + \sigma\int_0^t \exp(s)dB_s, \\ \exp(t)X_t &= X_0 + m\exp(t) - m + \sigma\int_0^t \exp(s)dB_s, \\ X_t &= \boxed{X_0\exp(-t) + m - m\exp(-t) + \sigma\int_0^t \exp(s-t)dB_s}. \end{aligned}$$

⌟

(b) Find $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t] := \mathbb{E}[(X_t - \mathbb{E}[X_t])^2]$.

Solution. For the expectation, we have:

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\left[X_0\exp(-t) + m - m\exp(-t) + \sigma\int_0^t \exp(s-t)dB_s\right] \\ &= X_0\exp(-t) + m - m\exp(-t) + \sigma\underbrace{\mathbb{E}\left[\int_0^t \exp(s-t)dB_s\right]}_{\text{deterministic, } 0} \\ &= \boxed{X_0\exp(-t) + m - m\exp(-t)}. \end{aligned}$$

For the variance, we hence have:

$$\begin{aligned} \mathrm{Var}[X_t] := \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] &= \mathbb{E}\left[\left(\sigma\int_0^t \exp(s-t)dB_s\right)^2\right] \\ &= \sigma^2\mathbb{E}\left[\left(\int_0^t \exp(s-t)dB_s\right)^2\right] = \sigma^2\int_0^t \exp\big(2(s-t)\big)ds \\ &= \sigma^2\left[\frac{\exp\big(2(s-t)\big)}{2}\right]_{s=0}^{s=t} = \sigma^2\left(\frac{1}{2} - \frac{\exp(-2t)}{2}\right) = \boxed{\frac{\sigma^2}{2}\big(1 - \exp(-2t)\big)}. \end{aligned}$$

⌟

Problem V.4. (Exercise 5.8 on [Øksendal]). Solve the (2-dimensional) stochastic differential equation:

$$dX_1(t) = X_2(t)dt + \alpha dB_1(t)$$
$$dX_2(t) = -X_1(t)dt + \beta dB_2(t)$$

where $\big(B_1(t), B_2(t)\big)$ is 2-dimensional Brownian motion and α, β are constants.
This is a model of a vibrating string subject to a stochastic force.

Solution. Here, we denote $X(t) := \big(X_1(t), X_2(t)\big)$ and $B(t) := \big(B_1(t), B_2(t)\big)$, so our differential equation becomes:

$$dX(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t)dt + \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} dB(t).$$

Here, we shall use the integrating factor that:

$$F(t) = \exp\left(t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n.$$

We note that the matrix has order 4, that is:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, we have the matrix exponential as:

$$F(t) = \begin{pmatrix} \sum_{n\in[0]_4} \frac{t^n}{n!} - \sum_{n\in[2]_4} \frac{t^n}{n!} & \sum_{n\in[1]_4} \frac{t^n}{n!} - \sum_{n\in[3]_4} \frac{t^n}{n!} \\ \sum_{n\in[3]_4} \frac{t^n}{n!} - \sum_{n\in[1]_4} \frac{t^n}{n!} & \sum_{n\in[0]_4} \frac{t^n}{n!} - \sum_{n\in[2]_4} \frac{t^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence, we have the solution as:

$$\begin{aligned} X(t) &= F(t)X(0) + F(t)\int_0^t F(-s) \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} B_t ds \\ &= \begin{pmatrix} X_1(0)\cos t + X_2(0)\sin t \\ -X_1(0)\sin t + X_2(0)\cos t \end{pmatrix} + \int_0^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos(-s) & \sin(-s) \\ -\sin(-s) & \cos(-s) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix} \\ &= \begin{pmatrix} X_1(0)\cos t + X_2(0)\sin t \\ -X_1(0)\sin t + X_2(0)\cos t \end{pmatrix} + \int_0^t \begin{pmatrix} \alpha\cos(t-s) & \beta\sin(t-s) \\ -\alpha\sin(t-s) & \beta\cos(t-s) \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix}. \end{aligned}$$

Hence, we have the solutions, respectively, as:

$$X_1(t) = \boxed{X_1(0)\cos t + X_2(0)\sin t + \alpha\int_0^t \cos(t-s)dB_1(s) + \beta\int_0^t \sin(t-s)dB_2(s)},$$
$$X_2(t) = \boxed{-X_1(0)\sin t + X_2(0)\cos t + -\alpha\int_0^t \sin(t-s)dB_1(s) + \beta\int_0^t \cos(t-s)dB_2(s)}.$$

⌟

Problem V.5. (Exercise 5.16 on [Øksendal]). For more general nonlinear stochastic differential equation of the form:

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t, \qquad X_0 = x, \tag{3}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ are given continuous (deterministic functions).

(a) Define the 'integration factor':

$$F_t = F_t(\omega) = \exp\left(-\int_0^t c(s)dB_s + \frac{1}{2}\int_0^t c^2(s)ds\right).$$

Show that (3) can be written as:

$$d(F_t X_t) = F_t \cdot f(t, X_t)dt. \tag{4}$$

Proof. Here, let's first derive dF_t using Itô formula with $dX_t = \frac{1}{2}c^2(t)dt - c(t)dB_t$:

$$dF_t = F_t\left(dX_t + \frac{1}{2}(dX_t)^2\right) = F_t\left(\frac{1}{2}c^2(t)dt - c(t)dB_t\right) + \frac{1}{2}F_t c^2(t)dt = F_t\big(c^2(t)dt - c(t)dB_t\big).$$

Therefore, we have the product rule resulting in:

$$\begin{aligned} d(F_t X_t) &= F_t dX_t + X_t dF_t + dF_t dX_t \\ &= F_t\big(f(t, X_t)dt + c(t)X_t dB_t\big) + X_t F_t\big(c^2(t)dt - c(t)dB_t\big) \\ &\quad + F_t\big(c^2(t)dt - c(t)dB_t\big)dB_t\big(f(t, X_t)dt + c(t)X_t dB_t\big) \\ &= F_t f(t, X_t)dt, \end{aligned}$$

as desired. □

(b) Now define:

$$Y_t(\omega) = F_t(\omega)X_t(\omega)$$

so that:

$$X_t = F_t^{-1}Y_t. \tag{5}$$

Deduce that equation (4) gets the form:

$$\frac{dY_t(\omega)}{dt} = F_t(\omega) \cdot f\big(t, F_t^{-1}(\omega)Y_t(\omega)\big), \qquad Y_0 = x. \tag{6}$$

Note that this is just a *deterministic* differential equation in the function $t \mapsto Y_t(\omega)$, for each $\omega \in \Omega$. We can therefore solve (6) with ω as a parameter to find $Y_t(\omega)$ and then obtain $X_t(\omega)$ from (5).

Proof. Here, from part (a), we have:

$$d\big(F_t(\omega)X_t(\omega)\big) = F_t f(t, X_t)dt = F_t f\big(t, F_t^{-1}(\omega)X_t(\omega)\big)dt,$$

which completes the proof when dividing both sides by dt. □

(c) Apply this method to solve the stochastic differential equation:

$$dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t, \qquad X_0 = x > 0,$$

where α is constant.

Solution. Here, we have the integrating factor as:

$$F_t = \exp\left(-\int_0^t \alpha dB_s + \frac{1}{2}\int_0^t \alpha^2 ds\right) = \exp\left(-\alpha\int_0^t dB_s + \frac{\alpha^2}{2}\int_0^t ds\right) = \exp\left(-\alpha B_t + \frac{\alpha}{2}t\right).$$

Then, by (b), let $Y_t := F_t X_t$, we have that:

$$\frac{dY_t}{dt} = \exp\left(-\alpha B_t + \frac{\alpha}{2}t\right)\cdot\frac{1}{\exp\left(-\alpha B_t + \frac{\alpha}{2}t\right)Y_t} = Y_t.$$

Hence, this becomes a trivial ODE, that is:

$$Y_t dY_t = dt, \qquad \text{and the solution is } Y_t = \sqrt{2t + Y_0^2}.$$

Therefore, we can deduce X_t as:

$$X_t = \boxed{\exp\left(\alpha B_t - \frac{\alpha}{2}t\right)\cdot\sqrt{2t + x^2}}.$$

┘

(d) Apply the method to study the solutions of the stochastic differential equation:

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t, \qquad X_0 = x > 0,$$

where α and γ are constants.
For what values of γ do we get explosion?

Solution. Here, we still have the integrating factor as:

$$F_t = \exp\left(-\int_0^t \alpha dB_s + \frac{1}{2}\int_0^t \alpha^2 ds\right) = \exp\left(-\alpha\int_0^t dB_s + \frac{\alpha^2}{2}\int_0^t ds\right) = \exp\left(-\alpha B_t + \frac{\alpha}{2}t\right).$$

Then, by (b), let $Y_t := F_t X_t$, we have that:

$$\frac{dY_t}{dt} = \exp\left(-\alpha B_t + \frac{\alpha}{2}t\right)\left(\exp\left(-\alpha B_t + \frac{\alpha}{2}t\right)Y_t\right)^\gamma = \exp\left(\left(-\alpha B_t + \frac{\alpha}{2}t\right)(1+\gamma)\right)Y_t^\gamma.$$

Again, this is still a separable ODE, and we have:

$$Y_t^{-\gamma}dY_t = \exp\left(\left(-\alpha B_t + \frac{\alpha}{2}t\right)(1+\gamma)\right)dt.$$

However, we note have a closed-form solution, and the solution is:

$$Y_t = \begin{cases} \left(\int_0^t \exp\left(\left(-\alpha B_s + \frac{\alpha}{2}s\right)(1+\gamma)\right)ds(1-\gamma)\right)^{\gamma-1} & \text{when } \gamma \neq 1 \\ \exp\left(\int_0^t \exp\left(\left(-\alpha B_s + \frac{\alpha}{2}s\right)\right)ds\right) & \text{when } \gamma = 1. \end{cases}$$

Hence, we have that:

$$X_t = \boxed{\begin{cases} \exp\left(\alpha B_t - \frac{\alpha}{2}t\right)\left(\int_0^t \exp\left(\left(-\alpha B_s + \frac{\alpha}{2}s\right)(1+\gamma)\right)ds(1-\gamma)\right)^{\gamma-1} & \text{when } \gamma \neq 1 \\ \exp\left(\alpha B_t - \frac{\alpha}{2}t\right)\exp\left(\int_0^t \exp\left(\left(-\alpha B_s + \frac{\alpha}{2}s\right)\right)ds\right) & \text{when } \gamma = 1. \end{cases}}$$

Note that the solution would explode when $\gamma > 1$. ⌟

Problem V.6. (Exercise 5.17 on [Øksendal]). **The Gronwall inequality**.
Let $v(t)$ be a nonnegative function such that:

$$v(t) \leq C + A\int_0^t v(s)ds \text{ for } 0 \leq t \leq T$$

for some constants C, A, where $A \geq 0$. Prove that:

$$v(t) \leq C\exp(At) \text{ for } 0 \leq t \leq T.$$

Hint: We may assume $A \neq 0$. Define $w(t) = \int_0^t v(s)ds$. Then $w'(t) \leq C + Aw(t)$. Deduce that:

$$w(t) \leq \frac{C}{A}\big(\exp(At) - 1\big)$$

by considering $f(t) := w(t)\exp(-At)$.

Proof. Consider that $w(t) = \int_0^t v(s)ds$, so by using Leibniz rule, its derivative is:

$$w'(t) = v(t) \leq C + A\int_0^t v(s)ds = C + Aw(t).$$

Then, we consider $f(t) := w(t)\exp(-At)$, and we take its derivative using the product rule:

$$f'(t) = w'(t)\exp(-At) - Aw(t)\exp(-At) = \exp(-At)\big(w'(t) - Aw(t)\big) \leq C\exp(-At).$$

Again, by the Leibniz rule and the previous inequality, while noting $f(0) = 0$, we have:

$$f(t) = \int_0^t f'(s)ds \leq \int_0^t C\exp(-As)ds = -\frac{C}{A}\big(\exp(-At) - 1\big).$$

Recall that $\exp(-At)$ is always positive, we can divide both sides by it:

$$w(t) = \frac{f(t)}{\exp(-At)} \leq \frac{-\frac{C}{A}\big(\exp(-At) - 1\big)}{\exp(-At)} = \frac{C}{A}\big(\exp(At) - 1\big).$$

Thus, we can extend the conclusion to $v(t)$, in which:

$$v(t) \leq C + Aw(t) = C + C\big(\exp(At) - 1\big) = C\exp(At),$$

which completes the proof. □

Problem V.7. Let $X(t)$ solve the Langevin equation:

$$dX(t) = -\mu X(t)dt + \sigma dB_t$$

and suppose that X_0 is a $\mathcal{N}\left(0, \frac{\sigma^2}{2\mu}\right)$ random variable. Show that:

$$\mathbb{E}[X(s)X(t)] = \frac{\sigma^2}{2\mu}e^{-\mu|t-s|}, \qquad t, s \geq 0.$$

Proof. Here, we first solve for the solution of Lagevin equation using the integrating factor:

$$F(t) = \exp(\mu t), \qquad \text{hence we have } dF(t) = \mu \exp(\mu t)dt.$$

Then, we have the product rule as:

$$\begin{aligned}
d\big(F(t)X(t)\big) &= F(t)dX(t) + X(t)dF(t) + dF(t)dX(t) \\
&= \exp(\mu t)\big(-\mu X(t)dt + \sigma dB_t\big) + \mu \exp(\mu t)X(t)dt \\
&= \sigma \exp(\mu t)dB_t,
\end{aligned}$$

and so the solution to the Lagevin equation is:

$$F(t)X(t) = F(0)X(0) + \int_0^t \sigma \exp(\mu s)dB_s = X(0) + \sigma \int_0^t \exp(\mu s)dB_s,$$

and hence we have:

$$X(t) = \exp(-\mu t)X(0) + \sigma \int_0^t \exp\big(\mu(s-t)\big)dB_s.$$

Then, we think about the expectation as:

$$\begin{aligned}
&\mathbb{E}[X(s)X(t)] \\
&= \mathbb{E}\left[\left(\exp(-\mu t)X(0) + \sigma \int_0^t \exp\big(\mu(u-t)\big)dB_u\right)\left(\exp(-\mu s)X(0) + \sigma \int_0^s \exp\big(\mu(u-s)\big)dB_u\right)\right] \\
&= \exp\big(-\mu(t+s)\big)\mathbb{E}[X^2(0)] + \exp(-\mu t)\sigma\mathbb{E}\left[X(0)\int_0^s \exp\big(\mu(u-s)\big)dB_u\right] \\
&\quad + \exp(-\mu s)\sigma\mathbb{E}\left[X(0)\int_0^t \exp\big(\mu(u-t)\big)dB_u\right] + \sigma^2\mathbb{E}\left[\int_0^t \exp\big(\mu(u-t)\big)dB_u \int_0^s \exp\big(\mu(u-s)\big)dB_u\right] \\
&= \exp\big(-\mu(t+s)\big)\cdot\frac{\sigma^2}{2\mu} + \exp(-\mu t)\sigma\mathbb{E}[X(0)]\mathbb{E}\left[\int_0^s \exp\big(\mu(u-s)\big)dB_u\right] \\
&\quad + \exp(-\mu s)\sigma\mathbb{E}[X(0)]\mathbb{E}\left[\int_0^t \exp\big(\mu(u-t)\big)dB_u\right] + \sigma^2\mathbb{E}\left[\int_0^t \exp\big(\mu(u-t)\big)dB_u \int_0^s \exp\big(\mu(u-s)\big)dB_u\right] \\
&= \exp\big(-\mu(t+s)\big)\cdot\frac{\sigma^2}{2\mu} + \sigma^2\mathbb{E}\left[\int_0^t \exp\big(\mu(u-t)\big)dB_u \int_0^s \exp\big(\mu(u-s)\big)dB_u\right].
\end{aligned}$$

Now, the main goal is to evaluate the last integral. Without loss of generality, we assume that $0 \le t \le s$:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \exp\big(\mu(u-t)\big)dB_u \int_0^s \exp\big(\mu(u-s)\big)dB_u\right] \\
&= \mathbb{E}\left[\int_0^t \int_0^s \exp\big(\mu(u-t)\big)\exp\big(\mu(v-s)\big)dB_v dB_u\right] \\
&= \int_\Omega \int_{[0,t]} \int_{[0,s]} \exp\big(\mu(u+v)-(t+s)\big)dB_v(\omega)dB_u(\omega)d\omega \\
&= \mathbb{E}\left[\int_0^t \exp\big(2\mu v - \mu(t+s)\big)dB_v(\omega) + \int_t^s dB_u(\omega)\right] \\
&= \exp\big(-\mu(t+s)\big)\int_0^t \exp(2\mu v)dv = \frac{1}{2\mu}\exp\big(-\mu(t+s)\big)\Big[\exp(2\mu v)\Big]_{v=0}^{v=t} \\
&= \frac{1}{2\mu}\exp\big(-\mu(t+s)\big)\big(\exp(2\mu t)-1\big).
\end{aligned}$$

When plugged in together, we have:

$$\begin{aligned}
\mathbb{E}[X(s)X(t)] &= \exp\big(-\mu(t+s)\big)\cdot\frac{\sigma^2}{2\mu} + \frac{\sigma^2}{2\mu}\exp\big(-\mu(t+s)\big)\big(\exp(2\mu t)-1\big) \\
&= \frac{\sigma^2}{2\mu}\exp\big(-\mu(s-t)\big).
\end{aligned}$$

Note that since $s \ge t$ is by our assumption, and it would otherwise be $t-s$, and we can conclude by $|t-s|$, which result in:

$$\mathbb{E}[X(s)X(t)] = \frac{\sigma^2}{2\mu}\exp\big(-\mu|t-s|\big),$$

as desired. □

Problem V.8. Prove that if $p \ge 2$ and $X \in \mathcal{V}([0,T])$, then:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t X_s dB_s\right|^p\right] \le C_p T^{\frac{p-2}{2}}\mathbb{E}\left[\int_0^T |X_s|^p ds\right]$$

for some constant $C_p > 0$ depending only on p.

Proof. First of all, we have Itô isometry that:

$$\mathbb{E}\left[\left|\int_0^t X_s dB_s\right|^2\right] = \mathbb{E}\left[\int_0^t |X_s|^2 ds\right].$$

Given the absolute value, we have non-negativity, and hence:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t X_s dB_s\right|^2\right] \le \mathbb{E}\left[\int_0^T |X_s|^2 ds\right].$$

This part is partially adapted from external source. Here, we consider the function φ:

$$\varphi(x) = |x|^p.$$

Here, we have:

$$\varphi'(x) = \operatorname{sgn}(x)p|x|^{p-1}, \text{ and } \qquad \varphi''(x) = p(p-1)|x|^{p-2} \text{ a.a.}$$

Note that $\int_0^t X_s dB_s =: M$ is a martingale, and we denote its supremum by M^*, and by Martingale representation theorem, it can be written as:

$$M^p = \int_0^T \operatorname{sgn}(M_s)p|M_s|^{p-1}dM_s + \frac{1}{2}\int_0^T p(p-1)|x|^{p-2}(dM_s)^2.$$

In particular, the expectation is:

$$\mathbb{E}[|M|^p] \le \frac{p(p-1)}{2}\mathbb{E}[|M^*|^{p-2}|M|^2].$$

Then, to utilize the Hölder inequality with $q = \frac{p}{p-2}$, we have:

$$\mathbb{E}[|M^*|^{p-2}|M|^2] \le \mathbb{E}[|M^*|^p]^{\frac{p-2}{p}}\mathbb{E}[|M|^p]^{\frac{p}{2}} \cdot T^{\frac{p-2}{p}\cdot\frac{p}{2}}$$

Then, we have:

$$\mathbb{E}[|M^*|^p] \le C_p T^{\frac{p-2}{2}}\mathbb{E}[|X_s|^p].$$

□

VI Problem Set 6

Problem VI.1. Let us consider the one-dimensional SDE:

$$dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dB_t, \qquad X_0 = x \in \mathbb{R}.$$

(a) Does this equation admit strong solutions?

Solution. Here, this equation admits strong solution. First, we denote:

$$b(t,x) = \sqrt{1+x^2} + \frac{1}{2}x \qquad \text{and} \qquad \sigma(t,x) = \sqrt{1+x^2}.$$

We can verify this by showing that it satisfies the existence and uniqueness theorem for SDEs.

- **Linear growth**: We note that:

$$\begin{aligned}|b(t,x)| + |\sigma(t,x)| &= \left|\sqrt{1+x^2} + \frac{1}{2}x\right| + \left|\sqrt{1+x^2}\right| \\ &\le \frac{1}{2}|x| + 2\left|\sqrt{1+x^2}\right| \le \frac{1}{2}(1+|x|) + 2(1+|x|) = \frac{5}{2}(1+|x|).\end{aligned}$$

- **Lipschitz condition**: We note that the derivative of $\sigma(t,x)$ is:

$$\left|\frac{d\sigma}{dx}(t,x)\right| = \frac{|x|}{\sqrt{1+x^2}} < \frac{|x|}{\sqrt{x^2}} = 1.$$

  Hence, $\sigma(t,x)$ must be Lipschitz, since if we assume that $|\sigma(t,x) - \sigma(t,y)| > |x-y|$, then by the Cauchy's mean value theorem, we have:

$$\frac{|\sigma(t,x) - \sigma(t,y)|}{|x-y|} > 1 \text{ which implies that there exists some } \xi \in [x,y] \text{ such that } \frac{d\sigma}{dx}(t,\xi) > 1,$$

  which is a contradiction, so we have:

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le 2|\sigma(t,x) - \sigma(t,y)| + \frac{1}{2}|x-y| \le \frac{5}{2}|x-y|.$$

- **Initial condition**: Note that $X_0 = x \in \mathbb{R}$ is a constant, which is independent of the Brownian motion, and $\mathbb{E}[|x|^2] = x^2 < \infty$.

Therefore, the equation satisfies the **existence and uniqueness theorem**. Hence, the equation admits strong solution. ⌟

(b) Let $Y_t = \log\left(\sqrt{1+X_t^2} + X_t\right)$. Find the SDE Y_t satisfied.

Solution. Here, we want to use the Itô formula, here we consider the function:

$$g(x) = \log\left(\sqrt{1+x^2} + x\right).$$

Here, we take the partial derivatives with respect to x for $g(x)$, where we note that:

$$\begin{aligned} g'(x) &= \frac{\frac{x}{\sqrt{1+x^2}}+1}{\sqrt{1+x^2}+x} = \frac{\frac{x}{\sqrt{1+x^2}}+1}{\sqrt{1+x^2}+x} \cdot \frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}-x} \\ &= \frac{x+\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}-x}{1+x^2-x^2} = \sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}}, \\ g''(x) &= -\frac{1}{2}(1+x^2)^{-\frac{3}{2}} \cdot (2x) = -x(1-x^2)^{-\frac{3}{2}}. \end{aligned}$$

Then, we have:

$$\begin{aligned} dY_t &= \frac{\partial}{\partial t} g(X_t)dt + \frac{\partial}{\partial x} g(X_t)dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} g(X_t)(dX_t)^2 \\ &= \frac{1}{\sqrt{1+X_t^2}} dX_t - \frac{1}{2}\frac{X_t}{(1+X_t^2)^{\frac{3}{2}}}(dX_t)^2 \\ &= \frac{1}{\sqrt{1+X_t^2}}\left[\left(\sqrt{1+X_t^2}+\frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dB_t\right] \\ &\quad - \frac{1}{2}\frac{X_t}{(1+X_t^2)^{\frac{3}{2}}}\left[\left(\sqrt{1+X_t^2}+\frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dB_t\right]^2 \\ &= dt + \frac{X_t}{2\sqrt{1+X_t^2}}dt + dB_t - \frac{1}{2}\frac{X_t}{(1+X_t^2)^{\frac{3}{2}}}\left(1+X_t^2\right)dt \\ &= dt + \frac{1}{2}\frac{X_t}{\sqrt{1+X_t^2}}dt + dB_t - \frac{1}{2}\frac{X_t}{\sqrt{1+X_t^2}}dt = dt + dB_t. \end{aligned}$$

Hence, Y_t satisfies that $\boxed{dY_t = dt + dB_t}$. ⌟

(c) Deduce an explicit solution for X_t.

Solution. To find the solution, we have:

$$Y_t = Y_0 + \int_0^t ds + \int_0^t dB_s = Y_0 + t + B_t.$$

Also, we note that:

$$Y_0 = \log(\sqrt{1+x^2}+x),$$

so we have:

$$Y_t = \log(\sqrt{1+x^2}+x) + t + B_t.$$

Then, we can write Y_t as function of X_t:

$$\log\left(\sqrt{1+X_t^2}+X_t\right) = \log(\sqrt{1+x^2}+x) + t + B_t.$$

By taking the exponential on both sides, we have:

$$\sqrt{1+X_t^2}+X_t = e^t e^{B_t} + (\sqrt{1+x^2}+x),$$

and by some arithmetic deductions, we get to that:

$$X_t = \boxed{\frac{\left(e^t e^{B_t} + (\sqrt{1+x^2}+x)\right)^2 - 1}{2\left(e^t e^{B_t} + (\sqrt{1+x^2}+x)\right)}}.$$

⌟

Problem VI.2. Let us consider the one-dimensional SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = x \in \mathbb{R}.$$

Assume that b, σ satisfies the Lipschitz condition and linear growth condition. Moreover, assume σ is continuous differentiable with $|\sigma'(x)| \leq C < \infty$ and $\sigma(x) \geq \delta > 0$ for all $x \in \mathbb{R}$.

(a) Consider $f(x) = \int_0^x \frac{1}{\sigma(y)}dy$ and the process $Y_t = f(X_t)$. Find the SDE Y_t satisfies.

Solution. Here, by the **Leibniz rule**, we have that:

$$\frac{\partial f}{\partial x} = \frac{1}{\sigma(x)} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2} = -\frac{\sigma'(x)}{\left(\sigma(x)\right)^2}.$$

Then, we use Itô formula to derive that:

$$\begin{aligned}
dY_t &= \frac{\partial}{\partial t} f(X_t)dt + \frac{\partial}{\partial x} f(X_t)dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(X_t)(dX_t)^2 \\
&= \frac{1}{\sigma(X_t)}dX_t - \frac{1}{2}\frac{\sigma'(x)}{\left(\sigma(x)\right)^2}(dX_t)^2 \\
&= \frac{1}{\sigma(X_t)}\left[b(X_t)dt + \sigma(X_t)dB_t\right] - \frac{1}{2}\frac{\sigma'(x)}{\left(\sigma(x)\right)^2}\left[b(X_t)dt + \sigma(X_t)dB_t\right]^2 \\
&= \frac{b(X_t)}{\sigma(X_t)}dt + dB_t - \frac{1}{2}\frac{\sigma'(X_t)}{\left(\sigma(X_t)\right)^2}\left(\sigma(X_t)\right)^2 dt = \left(\frac{b(X_t)}{\sigma(X_t)} - \frac{1}{2}\sigma'(X_t)\right)dt + dB_t.
\end{aligned}$$

Hence, the SDE that Y_t satisfies is:

$$\boxed{dY_t = \left(\frac{b(X_t)}{\sigma(X_t)} - \frac{1}{2}\sigma'(X_t)\right)dt + dB_t}.$$

⌟

(b) Prove that, under the assumption in (a), the filtration $\mathcal{H}_t = \sigma(\{X_s\}_{0 \leq s \leq t})$ coincides with the natural filtration $\mathcal{F}_t = \sigma(\{B_s\}_{0 \leq s \leq t})$.

Proof. Here, we want to show the two inclusions for the filtrations.

- ($\mathcal{H}_t \subset \mathcal{F}_t$:) Note that by definition:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = x \in \mathbb{R},$$

  where b, σ satisfies the Lipschitz condition and linear growth condition. Also we have $X_0 = x \in \mathbb{R}$ independent of B_t in which $\mathbb{E}[|x|^2] = x^2 < \infty$. Hence, the SDE satisfies the existence and uniqueness theorem, and so X_t is adapted to $\sigma(\{B_s\}_{0 \le s \le t})$, and hence $\mathcal{H}_t \subset \mathcal{F}_t$.

- ($\mathcal{F}_t \subset \mathcal{H}_t$:) Here, recall from part (a), we have:

$$dB_t = \left(\frac{b(X_t)}{\sigma(X_t)} - \frac{1}{2}\sigma'(X_t) \right) dt + dY_t.$$

  Clearly, 1 satisfies the linear growth and Lipschitz condition, and we need to verify the first part, in which we denote:

$$\varphi(x) = \frac{b(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x).$$

  For the **linear growth** condition, we have that:

$$|\varphi(x)| \le \frac{|b(x)|}{|\sigma(x)|} + \frac{1}{2}|\sigma'(x)| \le \frac{B(1+|x|)}{\delta} + \frac{1}{2}C \le \left(\frac{B}{\delta} + \frac{C}{2} \right)(1+|x|).$$

  Note that we do **not** need Lipschitz condition, since we just need existence of a strong solution so that $\mathcal{F}_t$ is $\mathcal{M}_t := \sigma(\{Y_s\}_{t \le s})$-adapted.
  Also, note that f is monotonic, so it is injective, hence admitting a left-inverse f^{-1}. Note that σ is measurable, f is also measurable, so does the left-inverse f^{-1}. Hence, $\mathcal{M}_t = \mathcal{H}_t$, and so $\mathcal{F}_t \subset \mathcal{H}_t$.

With both inclusions, we have $\mathcal{H}_t = \mathcal{F}_t$, as desired. □

VII Problem Set 7

Problem VII.1. (Exercise 7.1 on [Øksendal]). Find the generator of the following Itô diffusions:

(a) $dX_t = \mu X_t dt + \sigma dB_t$ (The Ornstein-Uhlenbeck process), where $B_t \in \mathbb{R}$, and μ, σ are constants.

Solution. Here, let $f \in C_0^2(\mathbb{R})$ be arbitrary, and we write the process as:

$$dX_t = \underbrace{\mu X_t}_{b(X_t)} dt + \underbrace{\sigma}_{\sigma(X_t)} dB_t,$$

and we have the infinitesimal generator as:

$$Af(x) = \mu x \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} = \boxed{\mu x f'(x) + \frac{1}{2}\sigma^2 f''(x)}.$$

⌟

(b) $dX_t = rX_t dt + \alpha X_t dB_t$ (The geometric Brownian motion), where $B_t \in \mathbb{R}$, and r, α are constants.

Solution. Again, let $f \in C_0^2(\mathbb{R})$ be arbitrary, and we write the process as:

$$dX_t = \underbrace{rX_t}_{b(X_t)} dt + \underbrace{\alpha X_t}_{\sigma(X_t)} dB_t,$$

and we have the infinitesimal generator as:

$$Af(x) = rx \frac{\partial f}{\partial x} + \frac{1}{2}(\alpha x)^2 \frac{\partial^2 f}{\partial x^2} = \boxed{rxf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x)}.$$

⌟

(c) $dY_t = rdt + \alpha Y_t dB_t$, where $B_t \in \mathbb{R}$, and r, α are constants.

Solution. Once again, let $f \in C_0^2(\mathbb{R})$ be arbitrary, and we write the process as:

$$dY_t = \underbrace{r}_{b(Y_t)} dt + \underbrace{\alpha Y_t}_{\sigma(Y_t)} dB_t,$$

and we have the infinitesimal generator as:

$$Af(x) = r \frac{\partial f}{\partial x} + \frac{1}{2}(\alpha x)^2 \frac{\partial^2 f}{\partial x^2} = \boxed{rf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x)}.$$

⌟

(d) $dY_t = \begin{pmatrix} dt \\ dX_t \end{pmatrix}$, where X_t is as in (a).

Solution. While again, let $f \in C_0^2(\mathbb{R}^2)$ be arbitrary, and we write the process as:

$$dY_t = \begin{pmatrix} dt \\ \mu X_t dt + \sigma dB_t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ \mu X_t \end{pmatrix}}_{b(X_t)} dt + \underbrace{\begin{pmatrix} 0 \\ \sigma \end{pmatrix}}_{\sigma(X_t)} dB_t,$$

and we have the infinitesimal generator as:

$$Af(x_1, x_2) = \boxed{\frac{\partial f}{\partial x_1} + \mu x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x_2^2}}.$$

⌟

(e) $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1} \end{pmatrix} dB_t$, where $B_t \in \mathbb{R}$.

Solution. Even again, let $f \in C_0^2(\mathbb{R}^2)$ be arbitrary, and we write the process as:

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ X_2 \end{pmatrix}}_{b(X_1, X_2)} dt + \underbrace{\begin{pmatrix} 0 \\ e^{X_1} \end{pmatrix}}_{\sigma(X_1, X_2)} dB_t,$$

and we have the infinitesimal generator as:

$$Af(x_1, x_2) = \boxed{\frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}}.$$

⌟

Problem VII.2. (Exercise 7.2 on [Øksendal]). Find an Itô diffusion (*i.e.*, write down the stochastic differential equation for it) whose generator is the following:

(a) $Af(x) = f'(x) + f''(x)$; $f \in C_0^2(\mathbb{R})$.

Solution. Here, we reversely construct that:

$$\boxed{dX_t = dt + \sqrt{2} dB_t}.$$

⌟

(b) $Af(t, x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}$; $f \in C_0^2(\mathbb{R}^2)$, where c, α are constants.

Solution. Again, we reversely construct that:

$$\boxed{dX_t = \begin{pmatrix} 1 \\ cX_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} 0 \\ \alpha X_t^{(2)} \end{pmatrix} dB_t}.$$

⌟

(c) $Af(x_1,x_2) = 2x_2\frac{\partial f}{\partial x_1} + \log(1+x_1^2+x_2^2)\frac{\partial f}{\partial x_2} + \frac{1}{2}(1+x_1^2)\frac{\partial f}{\partial x_1^2} + x_1\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x_2^2}$, $f \in C_0^2(\mathbb{R}^2)$.

Solution. Once again, we reversely construct the $\sigma\sigma^\intercal$ matrix as:

$$\sigma\sigma^\intercal = \begin{pmatrix} 1+x_1^2 & x_1 \\ x_1 & 1 \end{pmatrix}.$$

Note that $\sqrt{1+x_1^2}\cdot\sqrt{1}$ is not the same as the diagonals, so σ must be a 2-by-2 matrix.
Suppose $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then we have:

$$\sigma\sigma^\intercal = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2, \end{pmatrix}$$

and we have a candidate of σ as:

$$\sigma = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}$$

Hence, we have:

$$\boxed{dX_t = \begin{pmatrix} 2X_t^{(2)} \\ \log\left(1+X_t^{(1)^2}+X_t^{(2)^2}\right) \end{pmatrix} dt + \begin{pmatrix} 1 & X_1 \\ 0 & 1 \end{pmatrix} dB_t}.$$

⌟

VIII Problem Set 8

Problem VIII.1. (Exercise 7.7 on [Øksendal]). Let B_t be Brownian motion on $\mathbb{R}^n$ starting at $x \in \mathbb{R}^n$ and let $D \subset \mathbb{R}^n$ be an open ball centered at x.

(a) Prove that the harmonic measure μ_D^x of B_t is rotation invariant (about x) on the sphere ∂D. Conclude that μ_D^x coincides with normalized surface measure σ on ∂D.

Proof. Without loss of generality, we suppose $x = 0$, since the harmonic measure and Brownian motion is translational invariant.
First, we want to show that the Brownian motion is invariant with rotations. Suppose $U \in \mathbb{R}^{n\times n}$ such that $UU^\intercal = \mathrm{Id}$. Hence, we have $\det U = 1$, and so when we have the change of variable $p \mapsto U \cdot p$, the probability measure is the same, so the rotation of a Brownian motion is still a Brownian motion.
Now, as we consider the definition of the harmonic measure of some $F \in \partial D$, we have that:

$$\mu_D^0(F) := Q^0[B_{\tau_D} \in F],$$

and consider a rotation centered at 0 as U, we then have:

$$\mu_D^0(U \cdot F) := Q^0[B_{\tau_D} \in U \cdot F] = Q^0[U \cdot B_{\tau_D} \in F] = Q^0[B_{\tau_D} \in F] = \mu_D^0(F),$$

as desired. Moreover, consider that the harmonic measure μ_D^x of B_t is rotational invariant about ∂D, for any point $d, d' \in \partial D$, we have that $\mu_D^x(d) = \mu_D^x(d')$ so the measure is uniformly distributed on the surface, and $\mu_D(\partial D) = 1$. Hence, it coincides with the normalized surface measure ω on ∂D. □

(b) Let ϕ be a bounded measurable function on a bounded open set $W \subset \mathbb{R}^n$ and define:

$$u(x) = \mathbb{E}^x[\phi(B_{\tau_W})] \qquad \text{for } x \in W.$$

Prove that u satisfies the classical mean value property:

$$u(x) = \int_{\partial D} u(y) d\sigma(y) \tag{7}$$

for all balls D centered at x with $\overline{D} \subset W$.

Proof. Here, we have $\phi \in L^1(W)$, so we have that:

$$u(x) = \int_{\partial D} u(y) d\mu_D^x(y) = \int_{\partial D} u(y) d\sigma(y),$$

since μ_D^x coincides with normalized surface measure σ. □

(c) Let W be as in (b) and let $w : W \to \mathbb{R}$ be *harmonic* in W, *i.e.*:

$$\Delta w := \sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} = 0 \qquad \text{in } W.$$

Prove that w satisfies the classical mean value property (7).

Proof. Here, recall Green's formula for Harmonic PDE, we set the problem as:

$$\begin{cases} \Delta w = 0, & \text{in } W, \\ w(x) = g(x), & \text{on } \partial W, \end{cases}$$

where we assume that $g(x)$ is bounded and measurable function on W.
Then, we have the model that $\mathbb{E}\left[g\left(B_t^x(\omega)\right)\right] = u(x)$, and naturally by (b), we have:

$$w(x) = \int_{\partial D} w(y) d\sigma(y).$$

□

Problem VIII.2. (Exercise 7.10 on [Øksendal]). Let X_t be the geometric Brownian motion:

$$dX_t = rX_t dt + \alpha X_t dB_t.$$

Find $\mathbb{E}^x[X_T \mid \mathcal{F}_t]$ for $t \leq T$ by different approaches.

(a) Using the Markov property.

Solution. Here, we use the **Markov property** so that:

$$\begin{aligned} \mathbb{E}^x[X_{t+(T-t)} \mid \mathcal{F}_t] &= \mathbb{E}^{X_t}[X_{T-t}] = \mathbb{E}[X_t] \cdot \mathbb{E}\left[\exp\left(\left(r - \frac{\alpha^2}{2}\right) t + \alpha B_t\right)\right] \\ &= X_t \exp\left(r(T-t)\right) = x \exp(rt) \exp\left(r(T-t)\right) = \boxed{x \exp(rT)}. \end{aligned}$$

┘

(b) Writing $X_t = xe^{rt} M_t$, where:

$$M_t = \exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right) \qquad \text{is a martingale.}$$

Solution. Here, we can write the expectation as:

$$\begin{aligned} \mathbb{E}^x[X_T \mid \mathcal{F}_t] &= \mathbb{E}^x[xe^{rT} M_T \mid \mathcal{F}_t] = xe^{rT}\mathbb{E}^x[M_T \mid \mathcal{F}_t] \\ &= x \exp(rT) \cdot M_t = \exp\left(r(T-t)\right) X_t = \boxed{x \exp(rT)}. \end{aligned}$$

┘

Problem VIII.3. (Exercise 8.1 on [Øksendal]). Let Δ denote the Laplace operator on $\mathbb{R}^n$.

(a) Write down, in terms of Brownian motion, a bounded solution g of the Cauchy problem:

$$\begin{cases} \dfrac{\partial g(t,x)}{\partial t} - \dfrac{1}{2}\Delta_x g(t,x) = 0, & \text{for } t > 0, x \in \mathbb{R}^n, \\ g(0,x) = \phi(x), \end{cases}$$

where $\phi \in C_0^2$ is given. (From general theory it is known that the solution is unique.)

Solution. Here, since $\phi \in C_0^2$, we know that ϕ is lower-bounded. Then, we consider the Itô diffusion:

$$dX_t = 0dt + \mathrm{Id}\, dB_t = dB_t.$$

Then, we have the generator of the Itô diffusion as:

$$Af = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial f}{\partial x_i^2} = \Delta_x f \qquad \text{for } f \in C^2(\mathbb{R}^n).$$

Hence, we can use **Feynman-Kac Formula** that:

$$g(t,x) = \mathbb{E}^x\left[\exp\left(-\int_0^t 0ds\right)\phi(X_t)\right] = \boxed{\mathbb{E}^x[\phi(B_t)]}.$$

┘

(b) Let $\psi \in C_b(\mathbb{R}^n)$ and $\alpha > 0$. Find a bounded solution u of the equation:

$$\left(\alpha - \frac{1}{2}\Delta\right) u = \psi \qquad \text{in } \mathbb{R}^n.$$

Prove that the solution is unique.

Proof. Here, we note that we want to create the same Itô diffusion:

$$dX_t = 0dt + \mathrm{Id}\, dB_t = dB_t.$$

Then, we have the generator of the Itô diffusion as:

$$Af = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial f}{\partial x_i^2} = \Delta_x f \qquad \text{for } f \in C^2(\mathbb{R}^n).$$

Then, we can use **Feynman-Kac Formula** that:

$$u(t,x) = \mathbb{E}^x\left[\exp\left(-\int_0^t \psi(X_s)ds\right)\right] = \mathbb{E}^x\left[\exp\left(-\int_0^t \psi(B_t)ds\right)\right],$$

and the solution is unique for a given initial condition by Feynman-Kac. □

Problem VIII.4. (Exercise 8.7 on [Øksendal]). Let X_t be a sum of Itô integrals of the form:

$$X_t = \sum_{k=1}^{n} \int_0^t v_k(s,\omega) dB_k(s),$$

where $(B_1, \cdots, B_n)$ is n-dimensional Brownian motion. Assume that:

$$\beta_t := \int_0^t \sum_{k=1}^{n} v_k^2(s,\omega) ds \to \infty \qquad \text{as } t \to \infty \text{ a.s.}$$

Prove that:

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2\beta_t \log\log \beta_t}} = 1 \qquad \text{a.s.}$$

Hint: Use the law of iterated logarithm.

Proof. Here, we consider the differential form:

$$dX_t = \sum_{k=1}^{n} v_k(s,\omega) dB_k(t).$$

Then, we note that this is a 1-dimensional Brownian motion, and the time change is:

$$\beta_t = \int_0^t \sum_{k=1}^{n} v_k^2(s,\omega) ds.$$

With this time change, we can consider:

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2\beta_t \log\log \beta_t}} = \limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = 1$$

almost surely by the law of iterated logarithm. □

Problem VIII.5. Find a solution to the following PDE:

(a)

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) + bx \dfrac{\partial}{\partial x} u(t,x) + \dfrac{\sigma^2}{2} \dfrac{\partial^2}{\partial x^2} u(t,x) = 0, & x \in \mathbb{R}, t \in (0,T); \\ u(T,x) = x, & x \in \mathbb{R}. \end{cases}$$

Solution. Here, we need to think about the process for the SDE, as follows:

$$dX_t = bX_t dt + \sigma dB_t,$$

so we have the Itô generator as:

$$Af = bx\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}.$$

However, note that x is not lower bounded, so we cannot use the **Feynman-Kac** backward equation, directly, but we can think of a mollifier for $\epsilon < 0$ that:

$$\begin{cases} \frac{\partial}{\partial t}u^{(\epsilon)}(t,x) + bx\frac{\partial}{\partial x}u^{(\epsilon)}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}u^{(\epsilon)}(t,x) = 0, & x \in \mathbb{R}, t \in (0,T); \\ u^{(\epsilon)}(T,x) = \max\{\epsilon, x\}, & x \in \mathbb{R}. \end{cases}$$

Here, we consider the solution as:

$$\mathbb{E}^x\left[\max\{\epsilon, X_T\}\right] \to \boxed{\mathbb{E}^x[X_T]},$$

where X_t is the solution to the OU process. ⌟

(b) What if the boundary condition was replaced by $u(T,x) = x^2$.

Solution. Then, we use the **backward Feynman-Kac Formula**, since x^2 is bounded below, so that:

$$u(t,x) = \boxed{\mathbb{E}^x\left[X_T^2\right]},$$

where X_t is the solution to the OU process. ⌟

Problem VIII.6. (Exercise 8.11 on [Øksendal]).

(a) Let $Y(t) = t + B(t)$ for $t \geq 0$. For each $T > 0$, find a probability measure $\mathbb{Q}_T$ on $\mathcal{F}_T$ such that $\mathbb{Q}_T \sim \mathbb{P}$ and $\{Y(t)\}_{t \leq T}$ is Brownian motion with respect to $\mathbb{Q}_T$. Use:

$$M_T d\mathbb{P} = M_t d\mathbb{P} \qquad \text{on } \mathcal{F}_t^{(n)}; t \leq T \text{ when } M \text{ is a martingale}$$

to prove that there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty$ such that:

$$\mathbb{Q}\mid_{\mathcal{F}_T} = \mathbb{Q}_T \qquad \text{for all } T > 0.$$

Solution. Here, we write the expression as:

$$dY(t) = \underbrace{1}_{a(t,\omega)} dt + dB(t),$$

and hence we have the martingale:

$$M_t = \exp\left(-\int_0^t dB_s - \frac{1}{2}\int_0^t ds\right) = \exp\left(-B(t) - \frac{1}{2}t\right),$$

and hence by **Girsanov theorem I**, we have:

$$d\mathbb{Q}(\omega) = \exp\left(-B(T) - \frac{1}{2}T\right) d\mathbb{P}(\omega),$$

while $Y(t)$ is a Brownian motion with respect to $\mathbb{Q}_T$ for $0 \le t \le T$.
Note that M_t is martingale, hence we can consider:

$$\mathbb{Q}_t \mid_{\mathcal{F}_s} = \mathbb{Q}_s \text{ for } t \ge s.$$

Hence, we can construct the measure from $\mathbb{Q}_t$ for a $t \ge 0$ in to $\mathbb{Q}$, as desired. ⌟

(b) Show that:

$$\mathbb{P}\left(\lim_{t\to\infty} Y(t) = \infty\right) = 1,$$

while:

$$\mathbb{Q}\left(\lim_{t\to\infty} Y(t) = \infty\right) = 0.$$

Why does not this contradict the Girsanov theorem?

Solution. Recall the **Law of Iterated Log**, we have:

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1,$$
$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 0.$$

Now, consider the probability measure of $\mathbb{P}$, we have:

$$\lim_{t\to\infty} \frac{B_t + t}{\sqrt{2t\log\log t}} \le \lim_{t\to\infty} \frac{t}{\sqrt{2t\log\log t}} \to \infty.$$

However, for the probability measure $\mathbb{Q}$, we have that:

$$\lim_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} \text{ not to } \infty \text{ a.s.}$$

Hence, we note that $\mathbb{P}$ and $\mathbb{Q}$ does not correspond, this is because $\mathbb{Q}$ is constructed from $T \to \infty$, but is does not align to the case for concrete T values. ⌟

Problem VIII.7. (Exercise 8.12 on [Øksendal]). Let:

$$dY(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}, \qquad t \le T.$$

Find a probability measure

Q on $\mathcal{F}_T^{(2)}$ such that $\mathbb{Q} \sim \mathbb{P}$ and such that:

$$\tilde{B}(t) := \begin{pmatrix} -3t \\ t \end{pmatrix} + \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$$

is Brownian motion with respect to $\mathbb{Q}$ and:

$$dY(t) = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} d\tilde{B}_1(t) \\ d\tilde{B}_2(t) \end{pmatrix}.$$

Solution. Here, we think about:

$$\tilde{B}(t) = \begin{pmatrix} a(t) + B_1(t) \\ b(t) + B_2(t) \end{pmatrix},$$

so that we have:

$$\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and hence $b = 1$ and $a = -3$.
Then, we will think about **Girsanov theorem I**, so we have:

$$M_t = \exp\left(-\int_0^t \begin{pmatrix} -3 \\ 1 \end{pmatrix} \begin{pmatrix} dB_1(t) & dB_2(t) \end{pmatrix} - \frac{1}{2}\int_0^t \begin{pmatrix} -3 \\ 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \end{pmatrix} ds \right) = \exp\left(3B_1(t) - B_2(t) - 5t\right),$$

which leads to the change in probability measure as:

$$d\mathbb{Q}(\omega) = \boxed{\exp\left(3B_1(T)(\omega) - B_2(T)(\omega) - 5T\right) d\mathbb{P}(\omega)}.$$

┘

Problem VIII.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $B = \{B_t\}_{t\geq 0}$ be a Brownian motion with respect to filtration $\{F_t\}_{t\geq 0}$.

(a) Let $b : \mathbb{R} \to \mathbb{R}$ be a bounded continuously differentiable function and x a fixed real number. Determine a new probability $\mathbb{Q}$ in $(\Omega, \mathcal{F})$, the process $W_t = B_t - \int_0^t b(B_s + x)ds$ is a Brownian motion when $0 \leq t \leq T$. Find the SDE $Y_t = x + B_t$ satisfied with respect to $\mathbb{Q}$, *i.e.*, with respect to W_t.

Solution. Here, we write the expression in terms of differential form:

$$dW_t = dB_t - b(B_t + x)dt.$$

Then, we use the **Girsanov theorem I** to obtain that:

$$M_t = \exp\left(-\int_0^t -b(B_s + x)dB_s - \frac{1}{2}\int_0^t b^2(B_s + x)ds \right).$$

Hence, with the change in measure, we have:

$$d\mathbb{Q}(\omega) = \exp\left(\int_0^T b(B_s + x)dB_s - \frac{1}{2}\int_0^T b^2(B_s + x)ds\right) d\mathbb{P}(\omega).$$

Then, for $Y_t = x + B_t$, we have:

$$dY_t = dB_t = dW_t + b(B_t + x)dt = \boxed{b(Y_t)dt + dW_t}.$$

⌟

(b) Let F be an antiderivative of b. Prove that $d\mathbb{Q} = Z_T d\mathbb{P}$ with:

$$Z_t = \exp\left(F(B_t + x) - F(x) - \frac{1}{2}\int_0^t [b'(B_s + x) + b^2(B_s + x)]ds\right).$$

Proof. Note that from (a), we have:

$$\begin{aligned} Z_T &= \exp\left(\int_0^T b(B_s + x)dB_s - \frac{1}{2}\int_0^T b^2(B_s + x)ds\right) \\ &= \exp\left(F(B_T + x) - F(x) - \int_0^T \frac{1}{2}b'(B_s + x)ds - \frac{1}{2}\int_0^T b^2(B_s + x)ds\right) \\ &= \exp\left(F(B_t + x) - F(x) - \frac{1}{2}\int_0^t [b'(B_s + x) + b^2(B_s + x)]ds\right), \end{aligned}$$

as desired. □

(c) Let Y be the solution of:

$$\begin{cases} dY_t = \tanh(Y_t)dt + dW_t, \\ Y_0 = x. \end{cases}$$

Find $\mathbb{E}[e^{\theta Y_t}]$ the Laplace transform of Y_t with respect to $\mathbb{P}$.

Solution. Here, we immediately notice that this is a great model to define another Brownian motion, namely:

$$\tilde{M}_T = \exp\left(-\int_0^T \tanh(Y_s)dW_s - \frac{1}{2}\int_0^T \tanh^2(Y_s)ds\right).$$

Hence, we have Y_t as a Brownian motion with measure:

$$\begin{aligned} d\mathbb{T} &= \exp\left(-\int_0^T \tanh(Y_s)dW_s - \frac{1}{2}\int_0^T \tanh^2(Y_s)ds\right) d\mathbb{Q} \\ &= \exp\left(-\int_0^T \tanh(Y_s)dW_s - \frac{1}{2}\int_0^T \tanh^2(Y_s)ds + \int_0^T b(B_s + x)dB_s - \frac{1}{2}\int_0^T b^2(B_s + x)ds\right) d\mathbb{P}. \end{aligned}$$

Then, we have the Laplace transformation as:

$$\mathbb{E}_{\mathbb{T}}[\exp(\theta Y_t)] = \exp\left(\frac{1}{2}\theta^2 t\right),$$

and hence, by the change of variable, we have:

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[\exp(\theta Y_t)] \\
&= \exp\left(\frac{1}{2}\theta^2 t - \int_0^T \tanh(Y_s)dW_s - \frac{1}{2}\int_0^T \tanh^2(Y_s)ds + \int_0^T b(B_s+x)dB_s - \frac{1}{2}\int_0^T b^2(B_s+x)ds\right) \\
&= \boxed{\begin{array}{l}\exp\left(\frac{1}{2}\theta^2 t - \int_0^T \tanh(Y_s)dB_s - \frac{1}{2}\theta^2 t - \int_0^T \tanh(Y_s)b(B_t+x)dt\right. \\ \left. -\frac{1}{2}\int_0^T \tanh^2(Y_s)ds + \int_0^T b(B_s+x)dB_s - \frac{1}{2}\int_0^T b^2(B_s+x)ds\right)\end{array}}.
\end{aligned}
$$

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