`AS.110.109 (01)`: Calculus 2 for Physical Science and Engineering

Recitation Sheets

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Contents

This document records the questions and solutions to the problems reviewed during the recitation for `AS.110.109 (01)` Calculus 2 in the Fall 2025 semester at the Johns Hopkins University.

- If you notice any error, please contact me via email (`sguo45@jhu.edu`).

Week 1 (8/26)

Problem I.1. Consider the following integration:

$$\int_1^2 \frac{1}{x} dx.$$

(a) Evaluate it using 4 squares in terms of Riemann sum.

Solution. Here, we pick the left end point, so we evaluate the following points:

$$x = 1, \frac{5}{4}, \frac{3}{2}, \text{ and } \frac{7}{4}.$$

Here, we will have 4 intervals each with length $1/4$, so the Riemann sum is:

$$\int_1^2 \frac{1}{x} dx \approx \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \boxed{\frac{319}{420}}.$$

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(b) Write down the Riemann sum as a limit for the integration.

Solution. Here, between 1 and 2 when split into n pieces, we will just have:

$$\int_1^2 \frac{1}{x} dx \approx \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{1 + i/n} = \sum_{i=0}^{n-1} \frac{1}{n+i}.$$

Hence, to write the integration as a limit, we have:

$$\int_1^2 \frac{1}{x} dx = \boxed{\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n+i}}.$$

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(c) Now, find the exact result using the fundamental theorem of calculus.

Solution. We will find the antiderivative of $\frac{1}{x}$, which is $\ln|x|$, hence, we have:

$$\int_1^2 \frac{1}{x} dx = \ln|3| - \ln|2| = \boxed{\ln(3/2)}.$$

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Problem I.2. What are some examples of function that are "Riemann integrable" and not "Riemann integrable?"

Solution. First, we note that a function that is not smooth can well be Riemann integrable, consider the function:

$$f(x) = 1 - |x|,$$

which is not smooth at $x = 0$, but we have:

$$\int_{-1}^{1} f(x)dx = 1.$$

When a function has some sort of vertical asymptotic behavior, it could not be Riemann integrable. Consider:

$$f(x) = \frac{1}{x},$$

and we are trying to integrate over $(0, 1)$, we have:

$$\int_{0}^{1} f(x)dx = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} f(x)dx = \lim_{\epsilon \to 0^+} \ln|x| \Big|_{\epsilon}^{1} = -\infty,$$

in which this function is not integrable over the region.
A common example of a non-"Riemann integrable" function is the Dirichlet function, defined as follows:

$$f(x) = \begin{cases} 1, & \text{when } x \in \mathbb{Q}, \\ 0, & \text{when } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Say we want to integrate over $(0, 1)$. Note that if we choose the partition differently, say $0, 1/3, 2/3$, the sum is 0, but if we pick $0, \sqrt{2}/2$, the sum is $\sqrt{2}/2$, and if we have infinitely many partitions, the result is **not** uniform and hence will not be valid.

If you continue to learn more math courses in the sequence, you will eventually see that the integration results in 0 through another integration approach called "Lebesgue integration."

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Week 2 (9/2)

Problem II.1. Evaluate the following indefinite integral using integration by substitution:

$$\int \tan x dx.$$

Solution. Here, we note that:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{1}{\cos x} d(\cos x) = \boxed{-\ln|\cos x| + C}.$$

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Problem II.2. Evaluate the following definite integral.

$$\int_1^2 e^{1/x} \cdot \frac{1}{x^2} dx.$$

Solution. We may use integration by substitution with $u = 1/x$:

$$\int e^{1/x} \cdot \frac{1}{x^2} dx = -\int e^{1/x} d(1/x) = -e^{1/x} + C.$$

Hence, by the fundamental theorem of calculus, we have:

$$\int_1^2 e^{1/x} \cdot \frac{1}{x^2} dx = -e^{1/x}\Big|_1^2 = \boxed{e - \sqrt{e}}.$$

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Problem II.3. Evaluate the following indefinite integration:

$$\int \cos(2t) \tan(t) dt.$$

Solution. We exhibit the trigonometric identities, that is:

$$\begin{aligned}\int \cos(2t)\tan(t)dt &= \int \left(2\cos^2(t) - 1\right)\frac{\sin(t)}{\cos(t)} dt = \int \left(2\sin(t)\cos(t) - \tan(t)\right) dt \\ &= \int \left(\sin(2t) - \tan(t)\right) dt = \boxed{-\frac{1}{2}\cos(2t) + \ln|\cos t| + C}.\end{aligned}$$

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Week 3 (9/9)

Problem III.1. Evaluate the following indefinite integral.

$$\int \frac{1}{x} \cos\left(\ln\left(\frac{1}{x}\right)\right) dx.$$

Solution. This problem requires doing integration by substitution twice, first, we let:

$$u = \frac{1}{x}, \qquad du = -\frac{1}{x^2} dx,$$

so we can actually substitute by:

$$\frac{du}{u} = -\frac{1}{x} dx,$$

and the original integral becomes:

$$-\int \cos\left(\ln(u)\right) \frac{1}{u} du.$$

Then, we use another integration by substitution with $v = \ln(u)$ and $dv = \frac{du}{u}$ to obtain the original integral as:

$$-\int \cos v dv = -\sin v + C = -\sin\left(\ln(u)\right) + C = \boxed{-\sin\left(\ln\left(\frac{1}{x}\right)\right) + C},$$

completing this antiderivative. ⌟

Problem III.2. Evaluate the following definite integral:

$$\int_0^{2\pi} (t - \pi)^3 \cos(t) dt.$$

Solution. Notice we can transform the definite integral by manipulating its bounds:

$$\int_0^{2\pi} (t - \pi)^3 \cos(t) dt = \int_{-\pi}^{\pi} t^3 \cos(t + \pi) dt.$$

We observe that:

$$(-t)^3 \cos(-t + \pi) = -t^3 \cos(t + \pi),$$

so the function within the integrand is odd, and we note that the upper bound and lower bound is symmetric about 0, this definite integral evaluates to $\boxed{0}$. ⌟

Problem III.3. Find the area of the region encapsulated by the following inequalities:

$$y \geq -\sin(x), \qquad y \leq x, \qquad \text{and} \qquad y \leq \pi - x.$$

Solution. For this problem, it would be at best to illustrate the graph of the region.

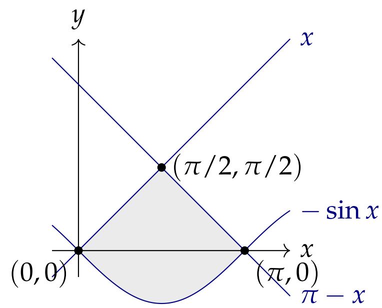

Figure III.1. Region encapsulated by the inequalities.

Hence, we can format the area as:

$$\int_0^{\pi/2}(x+\sin x)dx+\int_{\pi/2}^{\pi}(\pi-x+\sin x)dx=\left[\frac{x^2}{2}-\cos x\right]_0^{\pi/2}+\left[\pi x-\frac{x^2}{2}-\cos x\right]_{\pi/2}^{\pi}$$
$$=\frac{\pi^2}{8}+1+\pi^2-\frac{\pi^2}{2}+1-\frac{\pi^2}{2}+\frac{\pi^2}{8}=\boxed{\frac{\pi^2}{4}+2}.$$

⌟

Problem III.4. Let a cone has base radius r and height h, derive the volume of the cone as:

$$V=\frac{h\pi r^2}{3}.$$

Solution. We can construct the problem in terms of the volume of revolution the area between $y=rx/h$ and the positive real axis with respect to the x-axis from 0 to h. Hence, the integration is:

$$\int_0^h \pi\left(\frac{rx}{h}\right)^2 dx=\frac{\pi r^2}{h^2}\int_0^h x^2dx=\frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h=\frac{\pi r^2}{h^2}\cdot\frac{h^3}{3}=\frac{h\pi r^2}{3},$$

as desired. ⌟

Week 4 (9/16)

Problem IV.1. Recall from classical mechanics, we can model the gravitational force between two point masses of mass m_1 and m_2 separated by a distance r as:

$$F = \frac{Gm_1m_2}{r^2},$$

where $G \approx 6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ is the gravitational constant.

(a) Compute the gravitational force between the earth and a person of mass 100kg (replace this with something else if you would like to) at Baltimore with the following statistics:

mass of the Earth $\approx 5.97 \times 10^{24}$kg,

distance between Baltimore and center of the Earth $\approx 6.36 \times 10^6$m.

Note if this corresponds to the gravity of Earth constant that $g \approx 9.8\text{ms}^{-2}$.

Solution. The computation of the force is somewhat straightforward:

$$F = \frac{6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \times 5.97 \times 10^{24}\text{kg} \times 100\text{kg}}{(6.36 \times 10^6\text{m})^2} \approx \boxed{984\text{kgms}^{-2}}.$$

This result is quite close to the actual gravitational force computed by $g \approx 9.8\text{ms}^{-2}$. ⌟

(b) How much work needs to be done to lift this object to 3×10^6m above Baltimore against the gravitational force of Earth? Compare the result with using constant $g \approx 9.8\text{ms}^{-2}$.

Solution. We can write the force that the object has at height h as:

$$F(h) = \frac{6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \times 5.97 \times 10^{24}\text{kg} \times 100\text{kg}}{(6.36 \times 10^6\text{m} - h)^2}.$$

Then, the integration can be formatted as:

$$\int_0^{3\times 10^6\text{m}} F(h)dh.$$

Note that this integration is not so trivial, but what about doing a r-substitution to format it as distance from the center of Earth instead:

$$\int_0^{3\times 10^6\text{m}} F(h)dh = \int_{6.36\times 10^6\text{m}}^{9.36\times 10^6\text{m}} F(r + 3.36 \times 10^6\text{m})dr = \int_{6.36\times 10^6\text{m}}^{9.36\times 10^6\text{m}} \frac{3.98 \times 10^{16}\text{m}^3\text{kgs}^{-2}}{r^2}dr$$
$$= 3.98 \times 10^{16}\text{m}^3\text{kgs}^{-2} \cdot \left[-\frac{1}{r}\right]_{6.36\times 10^6\text{m}}^{9.36\times 10^6\text{m}} = \boxed{2.01 \times 10^9\text{kgm}^2\text{s}^{-2}}.$$

Note that if we assume the gravitation of the Earth being a constant, we would result in $3 \times 10^6\text{m} \times 980\text{kgms}^{-2} \approx 2.94 \times 10^9\text{kgm}^2\text{s}^{-2}$. This is larger since the gravitational force is weaker at higher height. ⌟

Problem IV.2. Evaluate the following indefinite integrations:

(a)

$$\int (3te^{-3t/2} + 2e^{-t/2})\,dt.$$

Solution. For this problem, we can use the linearity of integration to obtain that:

$$\begin{aligned}\int 3te^{-3t/2} + 2e^{-t/2}dt &= \int 3te^{-3t/2}dt + \int 2e^{-t/2}dt \\ &= -2te^{-3t/2} + 2\int e^{-3t/2}dt - 4e^{-t/2} + C \\ &= \boxed{-2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - 4e^{-t/2} + C}.\end{aligned}$$

⌟

(b)

$$\int \sin(5x)e^{-x}dx.$$

Solution. Here, we shall introduce a basic technique that you will see a lot over the course, we may integrate by parts twice and try to find some repetitive patterns.

$$\begin{aligned}\int \sin(5x)e^{-x}dx &= -\sin(5x)e^{-x} + \int 5\cos(5x)e^{-x}dx \\ &= -\sin(5x)e^{-x} + 5\left[-\cos(5x)e^{-x} - \int 5\sin(5x)e^{-x}dx\right] \\ 26\int \sin(5x)e^{-x}dx &= -\sin(5x)e^{-x} - 5\cos(5x)e^{-x} + C \\ \int \sin(5x)e^{-x}dx &= \boxed{-\frac{1}{26}\sin(5x)e^{-x} - \frac{5}{26}\cos(5x)e^{-x} + \tilde{C}}.\end{aligned}$$

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Week 5 (9/23)

Problem V.1. Solve the following definite integral by using trigonometric substitution.

$$\int_0^1 \frac{1}{\sqrt{x^2+1}}dx.$$

Solution. Here, we use the trigonometric substitution with $x = \tan t$, so we have $dx = \sec^2 t dt$, and we then have:

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x^2+1}}dx &= \int_0^{\pi/4} \frac{\sec^2 t}{\sqrt{\tan^2 t+1}}dt = \int_0^{\pi/4} \sec t dt \\ &= \int_0^{\pi/4} \frac{\sec t(\sec t+\tan t)}{\sec t+\tan t}dt \\ &= \int_1^{2/\sqrt{2}+1} \frac{1}{u}du = \ln\left|\frac{2}{\sqrt{2}}+1\right| - \ln|1| = \boxed{\ln\left(\frac{2\sqrt{2}+2}{2}\right)}.\end{aligned}$$

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Problem V.2. Now think about how to find the antiderivative using different approaches.

$$\int \arcsin x dx.$$

(a) Use integration by parts.

Solution. Consider the integration by parts with $u = \arcsin x$ and $dv = dx$, we have:

$$\begin{aligned}\int \arcsin x dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx &= x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}d(1-x^2) \\ &= x\arcsin x + \frac{1}{2}\cdot 2\sqrt{1-x^2} + C = \boxed{x\arcsin x + \sqrt{1-x^2}+C}.\end{aligned}$$

┘

(b) Use trigonometric substitution.

Solution. Consider the integration by substituting that $x = \sin t$ so $dx = \cos t dt$, and we have the integration as:

$$\begin{aligned}\int \arcsin x dx = \int t\cos t dt &= t\sin t - \int \sin t dt \\ &= t\sin t + \cos t + C = \arcsin x \cdot x + \cos(\arcsin x) + C \\ &= \boxed{x\arcsin x + \sqrt{1-x^2}+C},\end{aligned}$$

since we can recall that $\sin\theta = \frac{\text{OPP}}{\text{HYP}}$ and $\cos\theta = \frac{\text{OPP}}{\text{ADJ}}$. ⌟

Problem V.3. Factor the following polynomial into real, irreducible components.

$$f(x) = x^3 - 7x^2 + 16x - 12.$$

Solution. Recall from Pre-Calculus on the *Rational root test*:

Let the polynomial with integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$ satisfies that $p|a_0$ and $q|a_n$. ⌟

Hence, we can note that if the equation has a rational root, it must be one of:

$$r = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \text{ and } \pm 12.$$

By plugging in, one should notice that $x = 2$ is a root (one might also notice 3 is a root, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:

$$\frac{x^3 - 7x^2 + 16x - 12}{x - 2} = x^2 - 5x + 6,$$

Clearly, the right hand side is $(x-2)(x-3)$, so the polynomial factors into $\boxed{(x-2)^2(x-3)}$. ⌟

Problem V.4. Evaluate the following integration:

$$\int \frac{1}{x^3 - 1}\, dx.$$

Solution. We attempt to do the partial fraction for the integrand. Consider the roots of $x^3 - 1$ as $\zeta_3, \zeta_3^2, 1$, so we have:

$$\begin{aligned}
\frac{1}{x^3 - 1} = \frac{1}{(x-1)(x^2+x+1)} &= \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1} \\
&= \frac{Ax^2 + Ax + A + Bx^2 - Bx + Cx - C}{x^3 - 1} \\
&= \frac{(A+B)x^2 + (A - B + C)x + (A - C)}{x^3 - 1},
\end{aligned}$$

so we can solve for the system that:

$$\begin{cases} A + B = 0, \\ A - B + C = 0, \\ A - C = 1. \end{cases}$$

So we solve for that $A = \frac{1}{3}$, $B = -\frac{1}{3}$ and $C = -\frac{2}{3}$, so we have the integrand as:

$$\begin{aligned}
&\int \left(\frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)} \right) dx = \int \left(\frac{1}{3(x-1)} - \frac{2x+1+3}{6(x^2+x+1)} \right) dx \\
&= \frac{1}{3} \ln|x-1| - \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx - \frac{1}{2} \int \frac{1}{x^2+x+1} dx \\
&= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| - \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \\
&= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| - \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \\
&= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| - \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \arctan\left(\frac{2 \cdot \left(x - \frac{1}{2}\right)}{\sqrt{3}} \right) \\
&= \boxed{\frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| - \frac{\sqrt{3}}{3} \arctan\left(\frac{2\sqrt{3}x - \sqrt{3}}{3} \right)}.
\end{aligned}$$

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Week 6 (9/30)

Problem VI.1. Suppose that $p \in [1, \infty)$, verify the following claim about improper integrations:

$$\int_1^\infty x^{-p} dx$$

exists if and only if $p \neq 1$.

Solution. When $p = 1$, we have:

$$\int_1^\infty x^{-1} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} dx = \lim_{b \to \infty} \ln |x| \Big|_1^b = +\infty.$$

When $p > 1$, we have:

$$\int_1^\infty x^{-p} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \to \infty} \frac{1}{-(p-1)x^{p-1}} \Big|_1^b = \frac{1}{p-1} < +\infty.$$

Therefore, we can conclude that the improper integral converges if and only if $p \neq 1$ for all $p \in [1, \infty)$. ⌟

Problem VI.2. As known to most people today, $\sqrt{2}$ is an irrational number, which does not happens to be the case for the ancient Greeks. There happened to be various myths that someone is killed for discovering this $\sqrt{2}$ when most people believe all numbers have to be rational. We will try to investigate $\sqrt{2}$ through this problem.

(a) Find the limit of the sequence $\{x_i\}_{i=1}^\infty$ where $x_i = \sqrt{2} + \frac{1}{i}$.

Solution. This sequence *converges* to 1 as we have the limit:

$$\lim_{i \to \infty} x_i = \lim_{i \to \infty} \left(\sqrt{2} + \frac{1}{i} \right) = \sqrt{2} + \lim_{i \to \infty} \frac{1}{i} = \sqrt{2}.$$

⌟

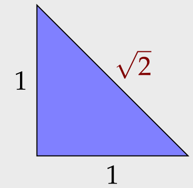
(b) Find the limit of the sequence $\{x_i\}_{i=1}^\infty$ where $x_i = \sqrt{2} + \frac{1}{i^2}$, does this vary from the previous part?

Solution. This sequence still *converges* to 1 as we have the limit:

$$\lim_{i \to \infty} x_i = \lim_{i \to \infty} \left(\sqrt{2} + \frac{1}{i^2} \right) = \sqrt{2} + \lim_{i \to \infty} \frac{1}{i^2} = \sqrt{2}.$$

⌟

The discovery of $\sqrt{2}$ can be constructed by the hypotenuse of a isosceles right triangle of side length 1.



However, when the Greeks were studying this, they first tried to write out $\sqrt{2}$ using a recursive definition, namely, the *continued fraction*:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

(c) Find a sequence of rational numbers using *continued fraction* that converges to $\sqrt{2}$.

Solution. We can truncate the fraction as we can write the sequence as follows:

$$\left\{1, 1 + \frac{1}{2} = \frac{3}{2}, 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}, 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{17}{12}, \cdots \right\}.$$

To find the general form, we can actually write $1 + \sqrt{2}$ as the continued fraction:

$$\sqrt{2} + 1 = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Here, we note that we have:

$$2 + \frac{1}{\sqrt{2} + 1} = 2 + \frac{\sqrt{2} - 1}{2 - 1} = \sqrt{2} + 1,$$

so we can have this structure as follows:

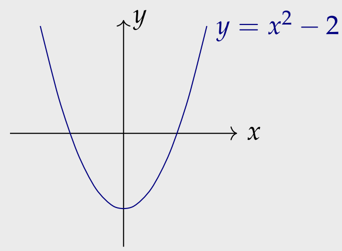
$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}.$$

Hence, for $x_i = \frac{p_i}{q_i}$, we should have:

$$x_{i+1} = \frac{q_i}{p_i} + 2 = \frac{q_i + 2p_i}{p_i},$$

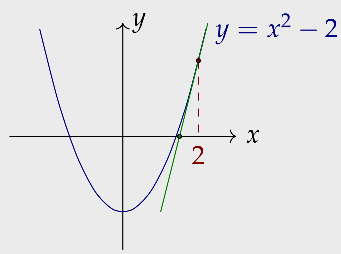
with the start being $x_1 = 2$. Note that this is the sequence converging to $\sqrt{2} + 1$, and the sequence with $\tilde{x}_i = x_i - 1$ will converge to $\sqrt{2}$. ⌟

The mathematics realm never lacks geniuses, and we will then think of another approach, called *Newton's method* to approximate $\sqrt{2}$ through a sequence. We consider $\sqrt{2}$ as the positive root of $x^2 - 2 = 0$, which can be illustrated in the following graph:



The Newton's method works as follows:

- Consider the first guess of the root as 2, we note $y(2) = 2 > 0$, so we consider the slope $y'(2) = 4$, so we assume that we draw the tangent line at 2 to get the next intersection at $2 - \frac{2}{4} = \frac{3}{2}$, which can be thought of as follows:



- Again, we consider the root as $\frac{3}{2}$ where we have $y\left(\frac{3}{2}\right) = \frac{1}{4}$, which is better, and we consider the slope $y'\left(\frac{3}{2}\right) = 3$, so we draw the tangent line again at $\frac{3}{2}$ so the new intersection is $\frac{3}{2} - \frac{1/4}{3} = \frac{17}{12}$.
- Then, we repeat this step further more for better approximations.

(d) Find a sequence of rational numbers using *Newton's Method* that converges to $\sqrt{2}$.

Solution. For this case, we can have the sequence as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - 2}{2x_i} = \frac{x_i}{2} + \frac{1}{x_i},$$

with x_1 being any arbitrary positive starting point. ⌟

Week 7 (10/7)

Problem VII.1. Recall that we have constructed the sequence $\{x_i\}_{i=1}^{\infty}$ as follows:

$$x_i = \sqrt{2} + \frac{1}{i}.$$

Show that this sequence converges to $\sqrt{2}$ by the definition of convergence (ϵ-N language).

Solution. Now, we consider any $\epsilon > 0$ to be arbitrary, we want to construct some critical point $N \in \mathbb{N}^+$ such that for all $i > N$, we have $|x_i - \sqrt{2}| < \epsilon$.
Here, we note that $|x_i - \sqrt{2}| = |\sqrt{2} + \frac{1}{i} - \sqrt{2}| = \frac{1}{i}$, since we know that given $\epsilon > 0$, there exists some integer M such that $\epsilon \cdot M > 1$, and hence, we have $\epsilon > \frac{1}{M}$, hence for all $i \geq M$, we have:

$$|x_i - \sqrt{2}| = \frac{1}{i} \leq \frac{1}{M} < \epsilon,$$

and thus we have shown that $x_i \to \sqrt{2}$ rigorously.
Aside, if you would not want to use the Archimedean property, we can also do a concrete construction (as how we did in class) $N = \left\lceil \frac{1}{\epsilon} \right\rceil + 1$ *so that we can still show the same argument.* ⌟

Problem VII.2. Given a sequence of functions $\{f_i\}_{i=1}^{\infty}$ defined as follows:

$$f_i(x) = x^i.$$

(a) Find the *pointwise* limit of the sequence of $f_i(x)$ for all $0 \leq x < 1$.
Hint: Consider that for each $0 \leq x_0 < 1$, find the limit of the sequence $\{f_n(x_0)\}_{n=1}^{\infty}$.

Solution. Given $0 \leq x_0 < 1$, we know that:

$$\lim_{i\to\infty} f_i(x_0) = \lim_{i\to\infty} x_0^i = 0,$$

hence the function converges pointwise to 0 on $[0, 1)$.
Of course, if you would like to argue this rigorously, consider any $\epsilon > 0$, *try to show that there exists some* $N \in \mathbb{N}^+$ *such that for all* $i \geq N$, *we have* $|x_0^i - 0| < \epsilon$. ⌟

(b) Find the *pointwise* limit of the sequence of $f_i(x)$ for $x = 1$.

Solution. Given $x = 1$, we know that:

$$\lim_{i\to\infty} f_i(x) = \lim_{i\to\infty} 1^i = 1,$$

hence the function converges pointwise to 1 at 1. ⌟

(c) Does the *pointwise* limit of the sequence of $f_i(x)$ exists for all $x > 1$?

Solution. No. The sequence diverges. Consider $x_0 > 1$, we have:

$$\lim_{i\to\infty} f_i(x_0) = \lim_{i\to\infty} x_0^i = \infty,$$

and hence the limit does not exist. ⌟

In fact, this sequence of functions is a classical example for mathematical analysis in distinguishing pointwise convergence and uniform convergence. If you find this interesting, think of what this sequence of functions will converge to and what are some good criterion to evaluate how well the function converges.

Problem VII.3. Consider if the following sequences $\{x_i\}_{i=1}^{\infty}$ converges or diverges.
(a) $x_i = e^{-i}\sin(i) + \cos(1/i)$.

Solution. Consider the limit:

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty} e^{-i}\sin(i) + \lim_{i\to\infty}\cos(1/i) = 0 + 1 = 1.$$ ⌟

(b) $x_i = \int_1^i \frac{1}{t^2}dt$.

Solution. Consider the limit, and it is an improper integral:

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty}\int_1^i \frac{1}{t^2}dt = \lim_{i\to\infty}\left[-\frac{1}{t}\right]_1^i = 1.$$ ⌟

(c) $x_i = \sum_{j=1}^{i}\frac{1}{\pi^j}$.

Solution. This is a geometric sequence with common ratio $\frac{1}{\pi} < 1$, so we have:

$$\lim_{i\to\infty} x_i = \sum_{i=1}^{\infty}\frac{1}{\pi^i} = \frac{1}{1-\frac{1}{\pi}} = \frac{\pi}{\pi - 1}.$$ ⌟

(d) $x_i = \prod_{j=1}^{i}\left(1 - \frac{1}{i+1}\right)$.

Solution. Notice that we are multiplying $0 < 1 - \frac{1}{i+1} < 1$ by each time so we have monotonicity and bounded as a nonnegative number, so the sequence converges. ⌟

Week 8 (10/14)

Problem VIII.1. Recall that we defined a sequence of functions last time, as $\{f_i\}_{i=1}^{\infty}$ by the following manner:

$$f_i(x) = x^i.$$

(a) Find the *pointwise* limit of the sequence of $f_i(x)$ for all $-1 < x < 0$.

Solution. From here, the problem got a little bit more interesting, x_0^i will be alternating between positive and negative for each consecutive terms, but we also have:

$$-(-x_0)^i \leq x_0^i \leq (-x_0)^i,$$

which since we have $-x_0 > 0$, we know that $-(-x_0)^i \to 0^-$ and $(-x_0)^i \to 0^+$. Therefore, by the squeeze theorem, we have $x_0^i \to 0$. ⌟

(b) Does the *pointwise* limit of the sequence of $f_i(x)$ exists for $x < -1$?

Solution. We can pick a subsequence $\{x_0^i\}_{i=2,4,6,\cdots}$, where we find this subsequence diverges, hence the sequence must diverge. ⌟

(c) Prove that the *pointwise* limit of the sequence of $f_i(x)$ does not exist for $x = -1$ (ϵ-N language).

Solution. Now, you should observe that $f_i(-1)$ is alternating between 1 and -1 throughout the time. For the sake of contradiction, we suppose that $(-1)^i$ converges to some $C \in \mathbb{R}$, for all $\epsilon > 0$, there must exist some $N \in \mathbb{N}^+$ such that for all $n \geq N$, we have $|(-1)^n - C| < \epsilon$.
Now, we pick $\epsilon = \frac{1}{3} > 0$, for any $(-1)^n$ and $(-1)^{n+1}$, one value will be -1 and another will be 1, there exists no C such that $|1 - C| < \frac{1}{3}$ and $|-1 - C| < \frac{1}{3}$, hence the limit does not exist. ⌟

Problem VIII.2. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence, and we define the partial sum $S_i = \sum_{j=1}^{i} x_i$, so we have $\{S_i\}_{i=1}^{\infty}$ as the series.

(a) If $\{x_i\}$ converges, does $\{S_i\}$ necessarily converge?
Give counterexample or explanation of why S_i converges.

Solution. No. Consider $x_i = \frac{1}{i}$, we shown in class that S_i diverges.
There could be a lot of counterexamples, so you can pick any. However, you might think of the example that $x_i = \frac{1}{i^2}$ so that S_i forms a geometric series which converges. However, as long as we have a counter example, the statement is false. ⌟

(b) If $\{S_i\}$ converges, does $\{x_i\}$ necessarily converge?
Give counterexample or just claim it is true.

Solution. Yes.
This would be involving an additional criterion that you will see very soon in class, i.e., *the Cauchy sequences. Since* $\{S_i\}$ *converges, it is Cauchy, so the differences between partial sums are small enough, so that the differences of consecutive sums will become a tool to show convergence for the elements* x_i. ⌟

Problem VIII.3. Geometric sequence is very useful tool applied in probability. Consider the following scenario.
Suppose you are flipping a fair coin (50% being head, 50% being tail) with your best friend, and you two can flip up to as many times.

(a) What is the probability that the first flip being a head?

Solution. $\frac{1}{2}$. ⌟

(b) What is the probability that the first head appears in the second flip?

Solution. This is the probability that the first is a tail while the second is head, so $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. ⌟

(c) What is the probability that the first head appears in the n-th flip?

Solution. This is the probability that the first $n-1$ flips are tails while the n-th flip is head, so the probability is $(\frac{1}{2})^{n-1} \cdot \frac{1}{2} = \frac{1}{2^n}$. ⌟

(d) What is the probability that there will never be a head?

Solution. We will use a weird approach to compute, the probability of having no head means the first appear of head is at the n-th flip but $n \to \infty$, so the probability is 0 since $\left\{\frac{1}{2^n}\right\}$ converges to 0. ⌟

Week 9 (10/21)

Problem IX.1. Given a sequence $\{x_n\}_{n=1}^{\infty}$ defined by:

$$x_n = e^{-n}\sin n.$$

(a) Is $\{x_n\}$ bounded or monotonic? Show your reasonings.

Solution. Here, we will provide a rigorous argument on how to show it strictly.

- **Monotonicity**: We notice that $e^{-n} > 0$ for all terms, and $\sin n > 0$ for $n \in \big(2k\pi, (2k+1)\pi\big)$, while $\sin n < 0$ for $n \in \big((2k-1)\pi, 2k\pi\big)$ for $k \in \mathbb{Z}$, so we know that $x_3 < 0$, $x_4 > 0$, and $x_7 > 0$ again, so it cannot be monotonic.
- **Boundedness**: Here, we notice that $|\sin n| \leq 1$ and $|e^{-n}| \leq \frac{1}{e}$ for $n \in \mathbb{N}^+$, this implies that:

  $$|e^{-n}\sin n| = |e^{-n}| \cdot |\sin n| \leq \frac{1}{e} \cdot 1 = \frac{1}{e}.$$

  Hence, x_i is bounded.

Therefore, the sequence is not monotonic but bounded. ⌟

(b) Does the sequence $\{x_n\}_{n=1}^{\infty}$ converge?

Solution. Yes.
We notice that for $n \in \mathbb{N}^+$, we have $-1 \leq \sin x \leq 1$, so we have:

$$-e^{-n} \leq x_n \leq e^{-n}.$$

It is noticeable that both lower and upper bound of x_n converges to 0. By the **Squeeze theorem**, we know that $\{x_n\}$ converges. ⌟

(c) Does $\sum_{n=1}^{\infty} e^{-n}\sin n$ converge? Show your reasoning.

Solution. For this integration, we can very naturally think of the integral test because the function is integrable in $[1, \infty)$, we note that $\sum_{n=1}^{\infty} e^{-n}\sin n$ is convergent if and only if $\int_1^{\infty} e^{-x}\sin x dx$ is convergent. Here, we work on this indefinite integral as follows:

$$\int_1^{\infty} e^{-x}\sin x dx = -\frac{1}{2}e^{-x}(\sin x + \cos x)\Big|_1^{\infty} = \frac{e^{-1}(\sin 1 + \cos 1)}{2} < +\infty.$$

Hence, the series converges. ⌟

Problem IX.2. The Fibonacci sequence is commonly defined using the following recursive relationship:

$$F(0) = 0, \qquad F(1) = 1, \qquad \text{and} \qquad F(n+2) = F(n+1) + F(n) \text{ for } n \geq 0.$$

The general formula of the sequence can be represented as the **Binet's formula**, described as follows:

$$F(n) = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Prove the Binet's formula by mathematical induction.

Solution. For this induction, we first need to check the base cases, *i.e.*, $F(0)$ and $F(1)$:

- $F(0) = \frac{1}{\sqrt{5}}(\alpha^0 - \beta^0) = 0$, which is good.
- $F(1) = \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1) = \frac{1}{\sqrt{5}}\frac{2\sqrt{5}}{2} = 1$, which is also valid.

Then, we proceed to the inductive step, we want to show that the formula still holds for $F(k+1)$ when $F(k)$ and $F(k-1)$ holds. That is, we now assume that for $k \geq 2$ such that:

$$F(k) = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) \qquad \text{and} \qquad F(k-1) = \frac{1}{\sqrt{5}}(\alpha^{k-1} - \beta^{k-1}).$$

Therefore, we consider the next term as:

$$\begin{aligned}
&F(k+1) = F(k) + F(k-1) = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) + \frac{1}{\sqrt{5}}(\alpha^{k-1} - \beta^{k-1}) \\
&= \frac{1}{\sqrt{5}}(\alpha^{k-1}(\alpha+1) - \beta^{k-1}(\beta+1)) = \frac{1}{\sqrt{5}}\left(\alpha^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - \beta^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right) \\
&= \frac{1}{\sqrt{5}}\left(\alpha^{k-1}\left(\frac{1+5+2\sqrt{5}}{4}\right) - \beta^{k-1}\left(\frac{1+5-2\sqrt{5}}{4}\right)\right) = \frac{1}{\sqrt{5}}(\alpha^{k-1}\alpha^2 - \beta^{k-1}\beta^2) = \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1}),
\end{aligned}$$

which completes the inductive step. Hence, this proves the Binet's formula. ⌟

Problem IX.3. Does this infinite product converge?

$$\prod_{n=1}^{\infty} \exp\left(\frac{2^n}{n!}\right).$$

Treat this as a sequence of elements.

Solution. Yes. Consider the partial product $p_i = \prod_{n=1}^{i} \exp\left(\frac{2^n}{n!}\right) = \exp\left(\sum_{n=1}^{i} \frac{2^n}{n!}\right)$. Note that:

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n} = 0,$$

hence the partial sum converges, so does the partial product. ⌟

Week 10 (10/28)

Problem X.1. (A Somewhat Hoax). In the field of advanced physics (like *String theory*), arguments were made on the following assumption:

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}.$$

Now, we will give an *illegal* proof below, and it is up to you to find where the incorrect assumption is made with the series rules that we have seen in class.

Illegal Proof. We will first show that the alternating series converges, *i.e.*:

$$\sum_{n=1}^{\infty} (-1)^n = \frac{1}{2}.$$

Here, we assume that the sum is S_0 (could be $\pm\infty$), and hence we have:

$$\begin{aligned} 2S_0 = S_0 + S_0 &= 1 + (-1) + 1 + (-1) + \cdots \\ &\qquad + 1 + (-1) + 1 + (-1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots, \end{aligned}$$

hence we have $S_0 = \frac{1}{2}$. Then, we define that $S_1 = \sum_{n=1}^{\infty} (-1)^n n$, and we have:

$$\begin{aligned} 2S_1 = S_1 + S_1 &= 1 + (-2) + 3 + (-4) + \cdots \\ &\qquad + 1 + (-2) + 3 + (-4) + \cdots \\ &= 1 + (-1) + 1 + (-1) + \cdots = S_0, \end{aligned}$$

and hence we have $S_1 = \frac{1}{4}$. Then, we consider the sum $S_2 = \sum_{n=1}^{\infty} n$ so we have:

$$\begin{aligned} S_2 - S_1 &= 1 + 2 + 3 + 4 + \cdots \\ &\quad + (-1) + 2 + (-3) + 4 + \cdots \\ &= 4 + 8 + 12 + \cdots = 4(1 + 2 + 3 + \cdots) = 4S_2, \end{aligned}$$

and hence we have $S_1 = -3S_2$, so we have $S_2 = -\frac{1}{12}$. :)

Solution. This issue for this problem is with the fact that one **cannot** change the order of the terms in a series unless the series converges absolutely.

For these series, one can easily show that the convergence is not absolute, so once cannot switch the order of terms.

As a side remark, it is still possible that if you have a series that is not absolutely convergent that is conditionally convergent, and by moving around some terms, the sum is still the same. However, in mathematical proofs, you cannot make a valid proof unless you show uniform convergence at the first step.

Hence, on a written assessment, if you want to show a series converges by switching orders of the terms, prove uniform convergence first! ⌟

Problem X.2. As we have started to explore about **Taylor series**, we will use Taylor series to compute the sum of certain series.

(a) Write down the Taylor series for e^x, $\sin x$, and $\cos x$ centered at 0.

Solution. One should either remember this, or should be able to derive this at an exam setting.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k,$$

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{k!} \sin^{(k)}(0)(x-0)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

$$\cos x = \sum_{k=0}^{\infty} \frac{1}{k!} \cos^{(k)}(0)(x-0)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

⌟

(b) Does $\sum_{k=0}^{\infty} \frac{1}{k!}$ converge? If so, give the explicit sum.

Solution. Yes.
We note that $\sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} 1^k = e^1 = \boxed{e}$. ⌟

(c) Does $\sum_{k=0}^{\infty} \frac{1}{(4k+1)!}$ converge? If so, give the explicit sum.

Solution. Yes.
We can first show that this series converges absolutely:

$$\sum_{k=0}^{\infty} \left| \frac{1}{(4k+1)!} \right| = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \sum_{k=1,5,9,\cdots} \frac{1}{k!} \leq \sum_{k=1}^{\infty} \frac{1}{k!} = e.$$

With absolute convergence, we can now freely move terms around, note we have:

$$\begin{array}{lcccccccc}
e^1 & = & +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\
e^{-1} & = & +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\
\sin 1 & = & & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\
\cos 1 & = & +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots
\end{array}$$

Since all these series converges *absolutely*, comparing vertically gives us that:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \boxed{\frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}}.$$

⌟

(d) Do the following series converge? If so, give the explicit sum.

(i) $\sum_{k=0}^{\infty} \frac{1}{(4k)!}$.

(ii) $\sum_{k=0}^{\infty} \frac{1}{(4k+2)!}$.

(iii) $\sum_{k=0}^{\infty} \frac{1}{(4k+3)!}$.

Solution. Yes, yes, and yes.

The verification that the series converges absolutely is basically the same as above and we will leave this as an exercise to the readers.

Recall the sums for e^1, e^{-1}, $\sin 1$, and $\cos 1$, we have for each of the infinite series composed of as a sum.

(i) $$\sum_{k=0}^{\infty} \frac{1}{(4k)!} = \boxed{\frac{e^1 + e^{-1}}{4} + \frac{\sin 1}{2}}.$$

(ii) $$\sum_{k=0}^{\infty} \frac{1}{(4k+2)!} = \boxed{\frac{e^1 + e^{-1}}{4} - \frac{\cos 1}{2}}.$$

(iii) $$\sum_{k=0}^{\infty} \frac{1}{(4k+3)!} = \boxed{\frac{e^1 + e^{-1}}{4} - \frac{\sin 1}{2}}.$$

┘

Week 11 (11/4)

Problem XI.1. For this problem, we will investigate another way of constructing a sequence of polynomials that approximates a function. Here, we consider the following function:

$$f(x) = \left| x - \frac{1}{2} \right|.$$

(a) Construct a sequence of Taylor polynomials centered at 0 for $f(x)$, what would be its radius of convergence?

Solution. It is not hard to notice that f is differentiable at 0 though (and it is infinitely differentiable, in fact), so we have $f'(0) = \frac{1}{2}$, $f'(0) = -1$ and all the rest derivatives are 0 at 0, so we have:

$$f_0(x) = \frac{1}{2} \qquad \text{and} \qquad f_i(x) = \frac{1}{2} - x \text{ for all } i \geq 1.$$

Note these polynomials agrees with $f(x)$ for $(-\infty, \frac{1}{2}]$, so the radius of convergence is $\boxed{\frac{1}{2}}$. ⌟

It is intuitive to ask if it's possible to have a polynomial approximation of $f(x)$ over $(0,1)$, *i.e.*, we want to have a sequence of "good functions" (like polynomials) that can converges to $f(x)$.
This is in fact, a well-known mathematical result, called **Bernstein polynomial**, which constructs a sequence of polynomial (which turns out to be infinitely smooth) that converges uniformly (which implies pointwise convergence) to a continuous function $f : [0,1] \to \mathbb{R}$, defined as:

$$f_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

(b) Note that $f(x)$ is continuous on $[0,1]$, write down $f_1(x)$ and $f_2(x)$.

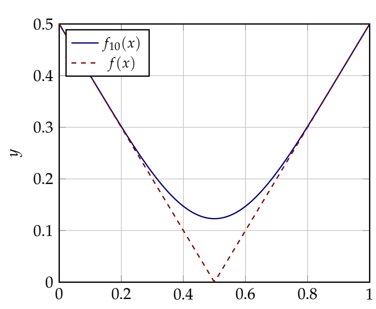
Solution.

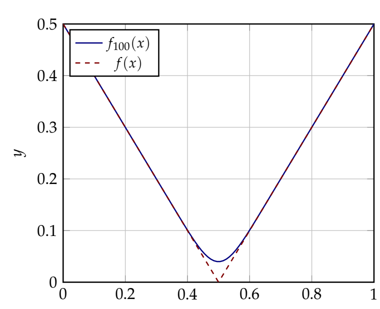
$$f_1(x) = \frac{1}{2}(1-x) + \frac{1}{2}x = \boxed{\frac{1}{2}},$$

$$f_2(x) = \frac{1}{2}(1-x)^2 + 0 + \frac{1}{2}x^2 = \boxed{x^2 - x + \frac{1}{2}}.$$

⌟

To satisfy our diligent readers, we provide the polynomials $f_{10}(x)$ and $f_{100}(x)$ with the following graphs:





Problem XI.2. (Applications of the Error function). e^{-x^2} is an important function, and the error function is defined as:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds.$$

Just to note, the additionally $\frac{2}{\sqrt{\pi}}$ is simply a normalizing constant, ensuring that $\operatorname{erf}(\infty) = 1$ (You will see a proof of this in *Calculus 3*, you can use this right now for granted).

(a) (Normal Distribution). Normal distribution captures the natural distribution and is an effectively tool when we have large number of trails for i.i.d. distributions. Given a normal distribution $X \sim N(0,1)$, we have its **probability density function** as:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The **cumulative density function** is defined as follows:

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(\xi) d\xi.$$

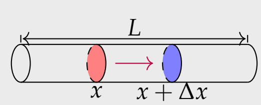
Express the cumulative density function of normal distribution using the error function.

Solution. This should be rather direct. Consider:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(\xi) d\xi = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi = \int_{-\infty}^{x/\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-u^2} \sqrt{2} du \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-u^2} du + \frac{1}{\sqrt{\pi}} \int_0^x e^{-u^2} du \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-u^2} du = \boxed{\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}. \end{aligned}$$

And therefore, as long as you can find a good numerate approximation, you can find the cumulative density of the normal distribution. ⌟

(b) (Heat Kernel). The study of PDEs can model how heat disperse on a rod.



Assume the rod has a negligible cross-sectional area and has infinite length, we model it's temperature $u(t,x)$ from half cold half hot. This system can be portrayed by the following PDE:

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} & \text{on } \mathbb{R}, \\ u(x,0) = 0 \text{ when } x \leq 0 & u(x,0) = 1 \text{ when } x > 0. \end{cases}$$

By utilizing the heat kernel and this initial condition, we can obtain the solution:

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi.$$

Write this solution in terms of the error function.

Solution. Here, we shall do a u-substitution with $\mu = \frac{x-\xi}{2\sqrt{t}}$ so we obtain taht:

$$\begin{aligned} u(x,t) &= \int_{\frac{x}{2\sqrt{t}}}^{-\infty} \frac{1}{\sqrt{4\pi t}} \exp(-\mu^2)(-2\sqrt{t}) d\mu = \int_{-\infty}^{\frac{x}{2\sqrt{t}}} \frac{1}{\sqrt{\pi}} \exp(-\mu^2) d\mu \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{\pi}} \exp(-\mu^2) d\mu + \int_0^{\frac{x}{2\sqrt{t}}} \frac{1}{\sqrt{\pi}} \exp(-\mu^2) d\mu \\ &= \boxed{\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)}. \end{aligned}$$

These steps requires very much efficiency for the readers to use u-substitutions and other fundamental properties of integrations. ⌟

Problem XI.3. Solve the initial value problem (IVP), specify the domain of solution:

$$\begin{cases} y' = (x \ln x)^{-1}, \\ y(e) = -6. \end{cases}$$

Solution. Here, we notice that this problem is separable, hence we can write:

$$dy = \frac{1}{x \log x} dx,$$
$$\int dy = \int \frac{1}{x \log x} dx.$$

Now, we evaluate the integral by substitution, *i.e.*, $u = \log x$ and $du = dx/x$, which give that:

$$y = \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C.$$

Eventually, we plug in the initial condition, that is $y(e) = -6$, giving us that:

$$-6 = \ln|\ln e| + C, \qquad \text{so } C = -6.$$

Therefore, the solution is:

$$y = \boxed{\ln|\ln x| - 6}.$$

Here, we note that $\ln(\cdot)$ has a valid domain over positive numbers, and the double $\ln(\cdot)$ functions with absolute value on the outer enforces that x must be either greater than 1 or in $(0,1)$, as $\ln(0)$ is undefined. Since our initial condition is e, and $e \in (1,\infty)$, the domain of the solution is $\boxed{(1,\infty)}$. ⌟

Week 12 (11/11)

Problem XII.1. Here, we will investigate the function $f(x) = \ln(x+1)$.

(a) Deduce the power series for $\ln(x+1)$ centered at 0.

Solution. Here, we take the derivative and note that:

$$\frac{d}{dx}\big(\ln(x+1)\big) = \frac{1}{x+1} = \frac{1}{1+x},$$

hence, as how we considered for geometric sequence, we have:

$$\frac{1}{1+x} = 1 + (-x) + (-x)^2 + \cdots = 1 - x + x^2 - x^3 + \cdots$$

Hence, we learned that the coefficients are just simply as $\frac{d^n}{dx^n}\frac{1}{1+x}\Big|_{x=0} = (-1)^n n!$. Hence, we can deduce the coefficients for $\log(x+1)$, that is:

$$\frac{d^n}{dx^n}\big(\log(x+1)\big)\Big|_{x=0} = (-1)^{n-1}(n-1)! \text{ for } n \geq 1.$$

Now, we can form the power series of $\ln(x+1)$ as:

$$\ln(x+1) \sim \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}}.$$

┘

(b) What is the interval of convergence for the power series?

Solution. Here, we notice that the derivative converges on the interval $(-1,1)$, hence we know that the power series of f shall also converge on $(-1,1)$, but we need to discuss the boundary case, respectively:

(i) For $x = 1$, we have the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

which converges by the alternating series test, since $\frac{(-1)^{n-1}}{n} \to 0$, so we have the sum convergent, and we have 1 within the interval of convergence.

(ii) For $x = -1$, we have the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges since it is a p-series with $p = 1$.

Therefore, the interval of convergence is $\boxed{(-1,1]}$.

┘

(c) Consider the differential equation given by:

$$(x+1)y'(x) = 1,$$

find the recurrence relationship and find the solution to the differential equations.

Solution. Suppose that the solution to the differential equation is formatted by (convergent on $(-1,1)$):

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

we compute the derivative as:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Here, we then $xy'(x) + y'(x) = 1$, so we can write it in terms of sums:

$$1 = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=1}^{\infty} \left((n+1)a_{n+1} + n a_n\right) x^n.$$

This implies that the recurrence relationship is that:

$$\begin{cases} a_1 = 1, \\ a_{n+1} = -\frac{n}{n+1} a_n & \text{for } n \geq 2. \end{cases}$$

From this recurrence relationship, we can notice that $a_i = (-1)^{i-1}\frac{1}{i}$, and thus the solution is the same as part (a), so the solution is $\boxed{\ln(x+1) + a_0}$, where we have to determine a_0 with initial condition. ⌟

(d) Solve the differential equation using the separation of variables.

Solution. Notice that the function is separable, so we can separate and integrate both side:

$$dy = \frac{dx}{x+1}.$$

$$y = \int dy = \int \frac{dx}{x+1} = \boxed{\ln|x+1| + C}.$$

which gives the same solution. ⌟

(e) Does the solution using power series and the separation of variables give the same result?
Hint: Consider the domain of convergence.

Solution. No. One can notice that $\ln(x+1) + C \neq \ln|x+1| + C$.
However, why is this the correct case? Notice that for our power series, the **interval of convergence** is $(-1,1)$, so as long as the solutions agree on $(-1,1)$, it does not violate anything. ⌟

Week 13 (11/18)

Problem XIII.1. Solve the following initial value problem (IVP):

$$\begin{cases} y' = y(y+1), \\ y(0) = 1. \end{cases}$$

Solution. We separate the problem as $\frac{dy}{y(y+1)} = dx$. Then, we use the partial fraction to obtain that:

$$\left(\frac{1}{y} - \frac{1}{y+1}\right) dy = dx.$$

Now, we can integrate both side to obtain that:

$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \int dx,$$

$$\ln\left|\frac{y}{y+1}\right| = \ln|y| - \ln|y+1| = x + C, \iff \frac{y}{y+1} = \tilde{C}e^x.$$

Now, we consider the left hand side as $\frac{y}{y+1} = \frac{y+1}{y+1} - \frac{1}{y+1} = 1 - \frac{1}{y+1}$, which rewrites into:

$$\frac{1}{y+1} = 1 - \tilde{C}e^x \iff y = \frac{1}{1-\tilde{C}e^x} - 1 = \frac{\tilde{C}e^x}{1-\tilde{C}e^x}.$$

By plugging in the initial condition, we obtain that:

$$1 = \frac{\tilde{C}}{1-\tilde{C}} \implies \tilde{C} = \frac{1}{2}.$$

Hence, the solution is $y = \boxed{\frac{e^x}{2-e^x}}$. For the domain of our solution, we note that e^x is continuous, but $2 - e^x$ could cause a zero denominator at $x = \ln 2$. Note that $\ln 2 > 0$, so the domain is $\boxed{(-\infty, \ln 2)}$. ⌟

Problem XIII.2. Solve for the general solution to the following differential equations.

(a) $2y' + y = 3t$.

Solution. Here, we first convert the equation to standard form, *i.e.*:

$$y' + \frac{1}{2}y = \frac{3}{2}t.$$

Hence, with $p(t) = 1/2$, the integration factor must be:

$$\mu(t) = \exp\left(\int_0^t p(s)ds\right) = \exp\left(\int_0^t \frac{1}{2}ds\right) = \exp\left(\frac{1}{2}t\right).$$

Now, we multiply the integration factor on both sides, giving that:

$$\frac{d}{dt}\left[e^{t/2}y\right] = y'e^{t/2} + \frac{1}{2}ye^{t/2} = \frac{3}{2}te^{t/2},$$
$$e^{t/2}y = \frac{3}{2}\int te^{t/2}dt = \frac{3}{2}\left[2te^{t/2} - \int 2e^{t/2}\right] = \frac{3}{2}\left[2te^{t/2} - 4e^{t/2} + C\right] = 3te^{t/2} - 6e^{t/2} + \tilde{C},$$
$$y = \boxed{\tilde{C}e^{-t/2} + 3t - 6}.$$

┘

(b) $y' + \ln(t)y = t^{-t}$.

Solution. Again, we consider the integrating factor here as:

$$\mu(t) = \exp\left(\int_e^t \ln s ds\right) = \exp(t\log t - t) = \frac{\exp(t\log t)}{e^t} = \frac{t^t}{e^t}.$$

Hence, we multiply the integration factor on both sides, giving that:

$$\frac{d}{dt}\left[\frac{t^t}{e^t}y\right] = y\frac{t^t}{e^t} + \frac{t^t}{e^t}\log ty = e^{-t},$$
$$\frac{t^t}{e^t}y = \int e^{-t}dt = -e^{-t} + C,$$
$$y = \boxed{-t^{-t} + Ct^{-t}e^t}.$$

┘

Problem XIII.3. Consider a second order differential equation be defined as follows:

$$y'' - y' - 2y = 0.$$

(a) Assume the solution is in the form of $y(t) = e^{at}$ where $a \in \mathbb{R}$ is some constants. Find some solutions for the differential equation in this form.
Hint: The solution might not be unique.

Solution. By assumption, we have $y(t) = e^{at}$, so $y'(t) = ae^{at}$, and $y''(t) = a^2e^{at}$, and by plugging into the differential equation, we have:

$$0 = a^2e^{at} - ae^{at} - 2e^{at} = e^{at}\left(a^2 - a - 2\right),$$

which implies that $a^2 - a - 2 = 0$, so $a = -1$ or 2, so the solutions in this form are $\boxed{e^{-t}}$ and $\boxed{e^{2t}}$. ┘

(b) Find all solutions to the given differential equations.

Solution. $\boxed{C_1e^{-t} + C_2e^{2t}}$ for all $C_1, C_2 \in \mathbb{R}$. ┘

Week 14 (12/2)

Problem XIV.1. Solve the following second order differential equations for $y = y(x)$:
(a) $y'' + y' - 132y = 0$

Solution. We find the characteristic polynomial as $r^2 + r - 132 = 0$, which can be trivially factorized into $(r - 11)(r + 12) = 0$, so with roots $r_1 = 11$ and $r_2 = -12$, we have the general solution as:

$$y(x) = \boxed{C_1 e^{11x} + C_2 e^{-12x}}.$$

(b) $y'' - 4y' = -4y$.

Solution. We turn the equation to the standard form $y'' - 4y' + 4 = 0$, and find the characteristic polynomial as $r^2 - 4r + 4 = 0$, which can be factorized into $(r - 2)^2 = 0$, so with roots $r_1 = r_2 = 2$ (repeated roots), we have the general solution as:

$$y(x) = \boxed{C_1 e^{2x} + C_2 x e^{2x}}.$$

(c) $y'' - 2y' + 3y = 0$.

Solution. We find the characteristic polynomial as $r^2 - 2r + 3 = 0$, which the quadratic formula gives:

$$r = \frac{2 \pm \sqrt{2^2 - 4 \times 3}}{2} = 1 \pm \mathrm{i}\sqrt{2}$$

so with roots $r_1 = 1 + \mathrm{i}\sqrt{2}$ and $r_2 = 1 - \mathrm{i}\sqrt{2}$, we would have the solution:

$$y(x) = C_1 e^{(1+\mathrm{i}\sqrt{2})x} + C_2 e^{(1-\mathrm{i}\sqrt{2})x}.$$

To obtain real solution, we apply Euler's identity:

$$y_1(x) = e^x\big(\cos(\sqrt{2}x) - \mathrm{i}\sin(\sqrt{2}x)\big) \text{ and } y_2(x) = e^x\big(\cos(-\sqrt{2}x) - \mathrm{i}\sin(-\sqrt{2}x)\big).$$

By the *principle of superposition*, we can linearly combine the solutions:

$$\widetilde{y_1}(x) = \frac{1}{2}(y_1 + y_2) = e^x \cos(\sqrt{2}x) \text{ and } \widetilde{y_2}(x) = \frac{1}{2}(y_2 - y_1) = e^x \sin(\sqrt{2}x).$$

Hence, the general solution as:

$$y(x) = \boxed{C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x)}.$$

Problem XIV.2. Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \sin^2(1-x)y = \cosh(x-1), \\ y(1) = e, \ \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique. Note that $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

Solution. The messy functions are deliberate. One cannot really find an elementary solution as well. Hence, we need to think on theorems.
Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution.
First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(2-x)] + \sin^2(1-x)y(2-x) &= y''(2-x) + \sin^2(x-1)y(2-x) \\ &= y''(2-x) + \sin^2\big(1-(2-x)\big)y(2-x) \\ &= y''(z) + \sin^2(1-z)y(z) \\ &= \cosh(z-1) = \frac{e^{z-1} + e^{-z+1}}{2} = \frac{e^{-(2-z)+1} + e^{(2-z)-1}}{2} \\ &= \cosh\big(x-1\big). \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) = e \text{ and } y'(1) = y'(2-1) = 0.$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.
Again, we observe the original initial value problem that:

$$\sin^2(1-x) \text{ and } \cosh(x-1) \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that $y(x) = y(2-x)$, so the solution is symmetric about $x = 1$, as desired. ⌟

Final Review

Review of contents for the semester.

1. Review on Elementary Integrations

Definite Integration

The motivation of integration is finding the (signed) area under the curve.

Area Under the Curve. Given a function $f(x)$ defined on an interval $[a, b]$, we consider the definite integral of f over $[a, b]$ as the signed area under the curve, denoted:

$$\int_a^b f(x)dx.$$

Note that this (signed) area can be positive, zero, or negative, the area enclosed by the curve but under the x-axis is considered negative area.

Properties of Integration. Consider functions f, g being well defined over $[a, b] \subset \mathbb{R}$ along with constant C, the definite integrals satisfy the following properties:

(i) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(ii) $\int_a^a f(x)dx = 0$.

(iii) $\int_a^b Cdx = C(b-a)$.

(iv) $\int_a^b \big(f(x) + cg(x)\big)dx = \int_a^b f(x)dx + c\int_a^b g(x)dx$.

(v) Consider $c \in [a, b]$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

(vi) If $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b f(x)dx$.

A numerical method to find the definite integration is by the **Riemann sum** technique.

Riemann Sum. Let $f(x)$ be defined over $[a, b]$, then Riemann sum considers the partition of $[a, b]$ into $a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ so that the approximation of integration is:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n-1} f(x_i) \cdot (x_{i+1} - x_i).$$

We say f is (Riemann) integrable on $[a, b]$ if for any partition, the limit exists:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n-1} f(x_i) \cdot (x_{i+1} - x_i).$$

If $f(x)$ is continuous over $[a, b]$ or has finitely many jump discontinuities, it is integrable over $[a, b]$.

Indefinite Integration

Alongside, we want to find an "inverse" of the derivatives operator, *i.e.*, we are finding the antiderivative as a function.

Fundamental Theorem of Calculus. Let $f(x)$ be continuous over $[a, b]$, and for $x \in [a, b]$, we define:

$$F(x) = \int_a^x f(t)dx$$

as the definite integral of f from a to the variable x.
Here, $F(x)$ is continuous on $[a, b]$ and is differentiable on (a, b) in the sense that $F'(x) = f(x)$.
Moreover, for F being any antiderivative function, we have:

$$\int_a^b f(x)dx = F(b) - F(a).$$

The function F here is the antiderivative function and is not unique, which is the result of a **indefinite integration**. The definite integration always gives a number and the indefinite integral gives a function.

Integration by u-Substitution

The process of finding antiderivatives is easy when the derivative turns out to the derivative of certain elementary functions, but integration by u-substitution allows integration of more functions.

The integration by u-substitution is motivated by the chain rule of differentiation.

Integration by u-substitution. Consider continuous functions $\varphi(x)$ and $g(x)$, where $\varphi(x)$ is continuous over $g([a, b])$ (the range of g) and g is continuously differentiable on $[a, b]$, then:

$$\int_a^b \varphi(g(x))g'(x)dx = \int_{g(a)}^{g(b)} \varphi(u)du.$$

Specifically, we replace $u = g(x)$ and so we replace $du = g'(x)dx$.

2. Applications of Integration

Enclosed Area

When looking for area, we would seek for a nonnegative quantity.

Area of Region. Suppose $f(x)$ and $g(x)$ are integrable over $[a, b]$ and $f(x) \geq g(x)$ over $[a, b]$, then the area enclosed by $f(x)$ and $g(x)$ over $[a, b]$ is:

$$\int_a^b (f(x) - g(x))dx.$$

Note that the monotonicity is not always guaranteed, in that case, the area between the region enclosed by $f(x)$ and $g(x)$ over $[a, b]$ can be generally be expressed as:

$$\int_a^b |f(x) - g(x)| dx,$$

but the evaluation of the integration over $|\cdot|$ shall be considered over each interval.

Volume of Revolution

The integration allows us to find the volume of 3-dimensional shapes.

Cross-Sectional Method. Consider the 3-dimensional solid S that lies between $x = a$ and $x = b$, and let $A(x)$ be the cross sectional area intersecting at x and suppose $A(x)$ is integrable over $[a, b]$, the volume of S is:

$$\int_a^b A(x) dx.$$

Note that this technique can be done over any cross-sectional area for x, y, or z.

For the cases of the volume of revolution, there will be two general methods to approach.

The Washer Method. Consider the solid constructed by revolving a region in 2-dimensional plane around an axis over $x = a$ to $x = b$, where the inner and outer distance to the axis is $f(x)$ and $g(x)$, the volume of revolution is:

$$\int_a^b \pi \big(g(x) - f(x)\big)^2 dx.$$

The Shell Method. Consider the solid constructed by revolving a region in 2-dimensional plane around an axis over from radius $r = a$ to $r = b$, where the height at each radius is modeled by $f(r)$, the volume of revolution is:

$$\int_a^b 2\pi r f(r) dr.$$

Work

The discussion of work relies heavily on the **Newton's Second Law** in classical mechanics:

$$F = m \cdot a = m \cdot \frac{d^2x}{dt^2}.$$

The basic definition of work is $W = F \cdot d$, *i.e.*, the force act upon the displacement. However, this does not account for the general case when the force could be varying.

Work. Assume an object is act by a force $f(x)$ at each position x as the object moves from $x = a$ to $x = b$, then the work done is:

$$\int_a^b f(x) dx.$$

When considering the average, we consider the integration but scaled by the inverse of the area.

Average Function. Consider $f(x)$ being integrable on $[a,b]$, the average of $f(x)$ over $[a,b]$ is:

$$\fint_a^b f(x)dx = \frac{1}{b-a} f(x)dx.$$

The mean value theorem associates the mean of an integration with finding an intermediate point for the averaging.

Mean Value Theorem. Consider $f(x)$ being continuous on $[a,b]$, then there always exists $c \in (a,b)$ such that:

$$\fint_a^b f(x)dx = f(c).$$

3. Integration Techniques

Integration by Parts

Integration by parts is inspired by the product rule and extends our capabilities of solving integrations.

Integration by Parts. Consider $f(x)$ and $g(x)$ being differentiable, we have:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx,$$

which can be equivalently formatted as:

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu.$$

As a side note, when dealing with indefinite integral, we often write:

$$\int udv = uv - \int vdu,$$

but if $f(x)g(x)$ altogether turns out to be a constant, then the term shall vanish.

Integration by Trigonometric Substitution

An important side mark about trigonometric functions is through the **Euler's formula**.

Euler's Formula. Given any $x \in \mathbb{R}$ and $\mathrm{i} := \sqrt{-1}$, we have:

$$e^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x).$$

One would be able to derive various trigonometric identities from here.

Especially with the Pythagorean identity in trigonometry, this allows us to do a trigonometric substitution with radicals.

Trigonometric Substitution.

Expression	Identity	Substitution	Range of θ
$\sqrt{a^2 - x^2}$	$\sin^2\theta + \cos^2\theta = 1$	$x = a\sin\theta$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\sqrt{a^2 + x^2}$	$1 + \tan^2\theta = \sec^2\theta$	$x = a\tan\theta$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\sqrt{x^2 - a^2}$	$1 + \tan^2\theta = \sec^2\theta$	$x = a\sec\theta$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Partial Fractions

The partial fraction decomposition decomposes a fraction of polynomial into sums of fractions in which the denominator are irreducible.

Partial Fraction Decomposition. Consider $f(x) \in K(\mathbb{R}[x])$ in which $f(x) = \frac{P(x)}{Q(x)}$, where we can have:

$$Q(x) = C(x+a_1)^{p_1} \cdot (x+a_2)^{p_2} \cdots (x+a_n)^{p_n} \cdot (x^2+b_1x+c_1)^{q_1} \cdots (x^2+b_mx+c_m)^{q_m},$$

then our guess of $f(x)$ can be written as the sum for different cases:

- For each $\frac{1}{(x+a)^p}$ term, we guess:

$$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_n}{(x+a)^n}.$$

- For each $\frac{1}{(x^2+bx+c)^q}$ term, we guess:

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_mx+C_m}{(x^2+bx+c)^m}.$$

Improper Integration

The improper integration expands to a large class of integrable functions.

Improper Integration, I. Suppose $f(x)$ is well defined over $[a, \infty)$, we defined the improper integral as:

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx,$$

if the limit exists (and is finite).
Suppose $f(x)$ is well defined over $(-\infty, b]$, we defined the improper integral as:

$$\int_{-\infty}^b f(x)dx = \lim_{t\to\infty} \int_{-t}^b f(x)dx,$$

if the limit exists (and is finite).

Note that when we consider the improper integration $f(x)$ over $(-\infty, \infty)$, we cannot just have a single limit, but rather:

$$\int_{-\infty}^\infty f(x)dx = \lim_{t\to\infty} \lim_{\tau\to\infty} \int_{-\tau}^t f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$$

to be finite and convergent for any convergence of t and τ. In this case, we say f is **integrable**.

A useful way of comparing is with monotone property.

Monotonicity of Integration Suppose $f(x)$ and $g(x)$ are well defined over $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, then:

- If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges.
- If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges.

The other type of improper integration deals with vertical asymptote cases.

Improper Integration, II. Suppose $f(x)$ is well defined over $[a, b)$, we defined the improper integral as:

$$\int_a^b f(x)dx = \lim_{t \nearrow b} \int_a^t f(x)dx,$$

if the limit exists (and is finite).
Suppose $f(x)$ is well defined over $(a, b]$, we defined the improper integral as:

$$\int_a^b f(x)dx = \lim_{t \searrow a} \int_t^b f(x)dx,$$

if the limit exists (and is finite).

Again, if f is only defined over $(a, b) \setminus \{c\}$, we consider:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

only if both integrations converges.

Handling on Infinity. Through the duration of mathematics, note that:

- $\infty + \infty = \infty$, $-\infty - \infty = -\infty$, $\infty \pm k = \infty$ for all $k \in \mathbb{R}$.
- However, $\infty - \infty$ tells us nothing, it is not ∞, not $-\infty$, and not any real number.

4. Sequences and Series

Sequence

A sequence is can be considered a order tuple.

Sequence. As sequence is an order list of numbers (*i.e.*, $\mathbb{R}^\mathbb{N}$) where we can denote it as $\{a_n\}_{n=1}^\infty$ assigned to positive integers ($\mathbb{N} := \{1, 2, \cdots\}$, or sometimes denoted $\mathbb{N}^+$).

Often time, there will be a provided formula or some recursive formulation of sequences using the notation.

Limit of Sequence A sequence $\{a_n\}_{n=1}^{\infty}$ has a limit $L \in \mathbb{R}$ if for every small $\epsilon > 0$, there exists a sufficiently large $N \in \mathbb{N}$ such that for every $n \geq N$:

$$L - \epsilon \leq a_n \leq L + \epsilon,$$

and we denote $\lim_{n\to\infty} a_n = L$.

We say that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges to ∞ if for any choice of $M \geq 0$, there exists some sufficiently large $N \in \mathbb{N}$ such that for all $n \geq M$, $a_n \geq M$, and we write $\lim_{n\to\infty} a_n = \infty$ (and this is not convergence).

Similarly, we say the sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$ if $\{-a_n\}_{n=1}^{\infty}$ diverges to ∞.

Limit Laws. It is important to recall some properties of limits consider the $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converging to A and B respectively with $\mathbb{C} \in \mathbb{R}$ being a fixed constant:

- $\lim_{n\to\infty}(a_n \pm c \cdot b_n) = A \pm c \cdot B$.
- $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$.
- Suppose $b_n \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.
- Suppose f being continuous at A, $\lim_{n\to\infty} f(a_n) = f(A)$.
- If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Recall our old friend from basic calculus.

Squeeze Theorem. Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$, and $a_n \leq c_n \leq b_n$ for all $n \geq N$ for some $N \in \mathbb{N}$. If:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

both converges, then $\lim_{n\to\infty} c_n = L$.

There are two important characteristic of Sequences.

Characteristics of Sequences. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence. It is:

- **monotonically increasing (or decreasing)** if $a_{n+1} \geq a_n$ (or $a_{n+1} \leq a_n$) for all n.
- **monotone** if it monotonically increase or monotonically decrease.
- **bounded above (or below)** if $a_n \leq M$ (or $a_n \geq M$) for all $n \in \mathbb{N}$.
- **bounded** if it is both bounded above and bounded below.

These two characters ensures us convergence in some sense.

Monotone Convergence Theorem. If a sequence is monotone and bounded, it converges and has a finite limit.

Series

The definition of series is parallel to the the definition of sequences.

Series. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, the series is defined to be the sum of the terms:

$$\sum_{n=1}^{\infty} a_n$$

as an infinite series.

Note that we can defined the partial sum as:

$$S_n = \sum_{n=1}^{n} a_i,$$

where the series converges if and only if the sequence of partial sums converges.

Geometric Series Consider the geometric series as:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots,$$

where $a \neq 0$, and here, this series converges if and only if $|r| < 1$.

In the sense of series, the convergence of a series implies the convergence of a sequence but not the converse.

Convergence Tests

Alternating Series Test. If $\sum_{n=1}^{\infty} a_n$ is a series whose terms alternate between nonnegative and nonpositive terms such that:

$$|a_{n+1}| \leq |a_n|,$$

then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

By the limit laws for sequences, some naturally extends to series.

Series Laws. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges:

- $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.
- For some fixed $c \in \mathbb{R}$, then $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

There also exists the comparison laws for the series.

Comparison Laws. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are nonnegative for all $n \geq N$ with some $N \geq 0$, suppose that $a_n \leq b_n$ for all $n \geq \tilde{N}$ with some $\tilde{N} \geq 0$, then:

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

With the idea of asymptotic behavior, we may conclude that:

Limit Comparison Test. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive for all $n \geq N$ with some $N \geq 0$ and:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = C.$$

for some $C > 0$ being finite, then

- either the series both converges,
- or the series both diverges

Integral, Ratio, and Root Test

It is possible to think of the Riemann sum and use that as a criterion of integral test.

Integral Test. Suppose that $\{a_n\}_{n=1}^{\infty}$ satisfies that $a_n = f(n)$ for some continuous, positive, decreasing function f defined over $[1, \infty)$, then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ is convergent.

There will also be a stronger convergence condition for series.

Absolute Convergence. $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.
$\sum_{n=1}^{\infty} a_n$ converges conditionally if it converges but not absolutely.

Then, we also consider the two tests based on the tail behaviors of the sequences.

Ratio Test. Suppose that $\{a_n\}_{n=1}^{\infty}$ satisfies that:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

- $\sum_{n=1}^{\infty} a_n$ converges absolute if $L < 1$.
- $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$.
- If $L = 1$, the test is not conclusive.

Root Test. Suppose that $\{a_n\}_{n=1}^{\infty}$ satisfies that:

$$\lim_{n \to \infty} |a_n|^{\frac{1}{n}} = L,$$

- $\sum_{n=1}^{\infty} a_n$ converges absolute if $L < 1$.

- $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$.
- If $L = 1$, the test is not conclusive.

Power Series

Then, we consider the concept of series into series of functions.

Power Series. A power series is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where we think of x as a variables, and the convergence could be dependent on the choice of x.

If we replace x by $x - a$, we effectively gets a power series centered at a.

Radius of Convergence. Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series, then exactly one of the following holds:

(i) The series converges for every $x \in \mathbb{R}$, in which we call f an analytic function.

(ii) The series only converges when $x = a$.

(iii) There exists a **radius of convergence** such that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges when $|x - a| < R$ and diverges when $|x - a| > R$.

A common power series is the case for geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

where $x \in (-1, 1)$, so the radius of convergence is 1.

This prepares us to use the calculus onto these series.

Term by Term Differentiation & Integration. Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has the radius of convergence $R > 0$, then:

- $f(x)$ is differentiable over $(a - R, a + R)$, where:

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\big(c_n (x-a)^n\big).$$

- $f(x)$ has antiderivatives over $(a - R, a + R)$, where:

$$f'(x) = \sum_{n=0}^{\infty} \int \big(c_n (x-a)^n\big) dx.$$

Taylor Series

The Taylor Series connects the power series representations with actual function.

Taylor Theorem. If $f(x)$ has a power series representation over $|x-a|<R$:

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n,$$

then we must have:

$$c_n=\frac{f^{(n)}(a)}{n!},$$

which is the Taylor series centered at a.

When $a=0$, such series is also called the Maclaurin series.

Since in reality, we cannot do a infinite sum, so we can analyze on remainder terms.

Taylor's Inequality. If $|f^{(n+1)}(x)|\le M$ for $|x-a|\le R$, then:

$$|R_N(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}\qquad \text{on } |x-a|\le R.$$

In particular, we can also try to formulate the remainders.

Remainder Forms. Suppose $f^{(n+1)}(x)$ is continuous on $|x-a|\le R$.

- **Integral Form**. We have:

$$R_N(x)=\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt\qquad \text{on } |x-a|\le R.$$

- **Leibniz Form**. Then there exists a $z\in(a,x)$ such that:

$$R_N(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$

5. Differential Equations

First Order ODEs

Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$). The order of the differential equation is the highest derivative with respect to t.

Separable ODEs. For ODEs in form $N(y)\frac{dy}{dt}=M(t)$, it can be separated as:

$$N(y)dy=M(t)dt.$$

Sometimes, the ODE might not be separable, so we can multiply an integrating factor to solve it.

Integrating Factor. For ODEs in form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int_0^t a(s)ds\right).$$

By multiplying $\mu(t)$ on both sides, the left hand side of the differential equation becomes $\frac{d}{dt}[\mu(t)y]$.

Typically, we focus on the linear operators here.

Linear Operator. In particular, for any x, y in the vector space and C as constant, a linear operator L satisfies that:

$$L[Cx + y] = CL[x] + L[y].$$

An example usage is on the half-life decay problem.

Half Life Problem. The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \ln 2.$$

Second Order ODEs

The second order ODEs opens up a wide range of differential equations for us to study.

Second Order Homogeneous ODEs with Constant Coefficients. Consider the ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is $r^2 + pr + q = 0$, with roots r_1 and r_2. We consider the general solution for different cases:

- If $r_1 \neq r_2$ are both real, the general solution is:

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}.$$

- If r_1 and r_2 are complex (but not real), by Euler's Formula, they can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1e^{\lambda t}\cos(\beta t) + c_2e^{\lambda t}\sin(\beta t).$$

- If $r_1 = r_2$ are repeated, then the solution is:

$$y(t) = c_1e^{rt} + c_2te^{rt}.$$

For second order differential equations, we have also introduced a theorem on existence and uniqueness.

Existence and Uniqueness Theorem. Consider initial value problem in form:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_1, y'(t_0) = y_2. \end{cases}$$

Let I be the interval containing t_0 such that $p(t)$, $q(t)$, and $g(t)$ are continuous. Then, there is a unique solution $y(t)$ and twice differentiable on the interval I.

Final Practices

Final practice problems and solutions.

1. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, evaluate the following integration.

$$\int_{-\infty}^{\infty} e^{ax^2} e^{-x^2} dx.$$

Specify the valid choices of $a \in \mathbb{R}$ in which the improper integration converges.

Solution.
We can rearrange the integration into:

$$\int_{-\infty}^{\infty} e^{ax^2} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(1-a)x^2} dx = \int_{-\infty}^{\infty} e^{-(\sqrt{1-a}x)^2} dx.$$

Therefore, we can use a u-substitution for $a < 1$ that:

$$u = \sqrt{1-a}x \qquad du = \sqrt{1-a}dx,$$

and by substitution, we obtain that:

$$\int_{-\infty}^{\infty} e^{ax^2} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-u^2} \cdot \frac{1}{\sqrt{1-a}} du = \frac{1}{\sqrt{1-a}} \int_{-\infty}^{\infty} e^{-u^2} du = \boxed{\frac{\sqrt{\pi}}{\sqrt{1-a}}}.$$

Then, we consider the case for $a = 1$, we have:

$$\int_{-\infty}^{\infty} 1 dx = \infty,$$

so the improper integration diverges.
When $a > 1$, we have e^{kx^2} for some $k > 0$, and thus we have:

$$x^2 \geq 0 \implies kx^2 \geq 0 \implies e^{kx^2} \geq 1,$$

so by monotonicity, we have:

$$\int_{-\infty}^{\infty} e^{kx^2} dx \geq \int_{-\infty}^{\infty} 1 dx = \infty,$$

so it also diverges.
Therefore, the valid choice of a in which the improper integration converges is $\boxed{a < 1}$.

2. Let k and m be positive integers, evaluate:

$$\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx.$$

Hint: Consider separately when $k = m$ and $k \neq m$.

Solution.
First, consider $k = m$, we have:

$$\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx = \int_0^\pi 0dx = 0.$$

Then, consider $k \neq m$, we have:

$$\begin{aligned}
&\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx \\
&= \int_0^\pi \big(\sin^2(kx) + \sin^2(mx) - 2\sin(kx)\sin(mx)\big) dx \\
&= \int_0^\pi \left(\frac{1-\cos(2kx)}{2} + \frac{1-\cos(2mx)}{2} - \cos\big((k-m)x\big) + \cos\big((k+m)x\big)\right) dx \\
&= x - \frac{\sin(2kx)}{4k} - \frac{\sin(2mx)}{4m} - \frac{\sin\big((k-m)x\big)}{k-m} + \frac{\sin\big((k+m)x\big)}{k+m}\Bigg|_{x=0}^{x=\pi} = \pi.
\end{aligned}$$

Therefore, we have computed that:

$$\int_0^\pi \big(\sin(kx) - \sin(mx)\big)^2 dx = \boxed{\begin{cases} 0, & \text{when } k = m, \\ \pi, & \text{when } k \neq m. \end{cases}}$$

3. Evaluate the following indefinite integration:

$$\int x \arcsin x dx.$$

Solution.

Here, we first do an integration by parts by picking $u = \arcsin x$ and $dv = xdx$, so we have $du = \frac{dx}{\sqrt{1-x^2}}$ and $v = \frac{x^2}{2}$, so integration by parts gives that:

$$\int x \arcsin x dx = \frac{x^2 \arcsin x}{2} - \int \frac{x^2}{2\sqrt{1-x^2}} dx.$$

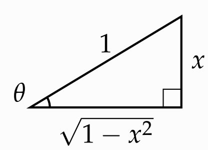
Now, our attention shall focus on the integral:

$$\begin{aligned}\int \frac{x^2}{2\sqrt{1-x^2}} dx &= \int \left(-\frac{1-x^2}{2\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \right) dx \\ &= -\frac{1}{2} \int \sqrt{1-x^2} dx + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \sqrt{1-x^2} dx + \frac{1}{2} \arcsin x + C.\end{aligned}$$

Eventually, we put our attention to the only integration again and do a trigonometric substitution for $x = \sin t$ so $dx = \cos t dt$:

$$\begin{aligned}\int \sqrt{1-x^2} dx = \int \sqrt{\cos^2 t} \cos t dt = \int \cos^2 t dx = \int \frac{1+\cos 2t}{2} dt \\ = \frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C.\end{aligned}$$

Here, we consider visualizing the right triangle with angle the angle θ and hypotenuse as 1, so we can visualize the case with this angle.



Therefore, we have $\sin(2\arcsin x) = 2\sin(\arcsin x)\cos(\arcsin x) = 2x\sqrt{1-x^2}$. Hence, as we put together all the components, we now have:

$$\begin{aligned}\int x \arcsin x dx &= \frac{x^2 \arcsin x}{2} - \frac{1}{2} \arcsin x + \frac{1}{2} \left(\frac{\arcsin x}{2} + \frac{2x\sqrt{1-x^2}}{4} \right) + C \\ &= \boxed{\frac{x^2 \arcsin x}{2} - \frac{\arcsin x}{4} + \frac{x\sqrt{1-x^2}}{4} + C}.\end{aligned}$$

4. Consider the three dimensional shape generate by rotating the function:

$$x = \sqrt{1-(y-2)^2} \qquad \text{for } x \geq 0$$

around the y-axis. Suppose $x = 0$ is ground level, and the sphere is filled up with liquid of density ρ and the gravitational acceleration is g. Compute the gravitational energy of the liquids in the sphere.

Solution.
At a given height y, we have the radius of revolution as $\sqrt{1-(y-2)^2}$, and the cross-sectional area is a circle, we have the potential energy:

$$\begin{aligned} E_p &= \int_1^3 h \cdot \rho \cdot A \cdot g dy = \int_1^3 y\rho g\pi\big(1-(y-2)^2\big)dy = \rho g\pi \int_1^3 y\big(1-(y-2)^2\big)dy \\ &= \rho g\pi \int_1^3 (y - y^3 + 4y^2 - 4y)dy = \rho g\pi \left[-\frac{y^4}{4} + \frac{4y^3}{3} - \frac{3y^2}{2}\right]_1^3 \\ &= \rho g\pi \left(-\frac{81}{4} + \frac{1}{4} + 36 - \frac{4}{3} - \frac{27}{2} + \frac{3}{2}\right) = \rho\pi g\left(4 - \frac{4}{3}\right) = \boxed{\frac{8}{3}\rho\pi g}. \end{aligned}$$

5. Let k be a positive integer, and given the series:

$$\sum_{n=1}^{\infty} \frac{e^n}{e^{kn}+n}.$$

Find the values of k such that the series converges.

Here, we consider using the ratio test:

$$a_n = \frac{e^n}{e^{kn}+n} \qquad \text{and} \qquad a_{n+1} = \frac{e^{n+1}}{e^{k(n+1)}+n+1}.$$

Therefore, the ratio as $n \to \infty$ is:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e(e^{kn}+n)}{e^{kn+k}+n+1}\right| = \lim_{n\to\infty} \frac{e^{kn+1}+en}{e^{kn+k}+n+1}.$$

Clearly, when $n \geq 1$, we have the denominator having larger power of the exponential, hence the limit converges to 0.
At $n = 1$, unfortunately, the limit approaches 1, which gives an inconclusive result.
However, consider when $k = 1$, we have:

$$a_n = \frac{e^n}{e^n+n} = 1 - \frac{n}{e^n+n} \to 1,$$

the terms do not tend to 0, hence it must be divergent.
Therefore, we must have $\boxed{k > 1}$ for the series to converge.

6. Evaluate the following series:
$$\sum_{k=0}^{\infty} \frac{1}{(2k)!}.$$
Hint: Find the Taylor expansion of e^x.

Solution.
Here, we can first verify that:
$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \leq \sum_{k=0}^{\infty} \frac{1}{k^2} < +\infty,$$
hence, we know that the series converges absolutely, so we are free to manipulate between the terms.
Then, we write down the Taylor expansion for e^x centered at 0, that is:
$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$
Note that we have an infinite radius of convergence for the Taylor expansion of e^x, since it is smooth, hence we can evaluate e^x at 1 and -1, that is:
$$e^1 = \sum_{k=0}^{\infty} \frac{1}{k!} \qquad \text{and} \qquad e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$
For the reader's simplicity, let's write down a few first terms:
$$\begin{aligned} e^1 &= +\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \\ e^{-1} &= +\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots \end{aligned}$$
It is not hard to realize that we can obtain the even terms simply by adding the two expressions:
$$e^1 + e^{-1} = 2\frac{1}{0!} + 2\frac{1}{2!} + 2\frac{1}{4!} + \cdots,$$
and hence we have:
$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \boxed{\frac{e^1 + e^{-1}}{2}}.$$

7. Solve the following differential equation:

$$y' + 3y = t + e^{-2t},$$

and discuss behavior of the solution as $t \to \infty$.

Solution.
Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$

By multiplying both sides with $\exp(3t)$, we obtain the equation:

$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$

Clearly, we observe that the left hand side is the derivative after product rule for ye^{3t} and the right hand side can be simplified as:

$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^t.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$ye^{3t} = \int te^{3t}dt + \int e^t dt = \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t}dt + e^t + C = \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^t + C.$$

Eventually, we divide both sides by e^{3t} to obtain that:

$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$

As $t \to \infty$, the solution diverges to ∞.

8. Given a second order differential equation with some $\lambda \in \mathbb{R}$ fixed:

$$\begin{cases} f''(x) - \lambda f(x) = 0, \\ f(0) = f(\pi) = 0. \end{cases}$$

It is know that the solution is nontrivial ($f \not\equiv 0$, *i.e.*, f must not be zero everywhere). Give the appropriate choice of λ and the corresponding solutions.

Solution.
For any $\lambda \in \mathbb{R}$, the characteristic equation is:

$$r^2 - \lambda = 0$$

Here, let's consider each cases for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

- When $\lambda > 0$, we would have $r = \pm\sqrt{\lambda}$, and thus the solution must be:

$$f(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}.$$

  It is not hard to see that since $e^{(\cdot)}$ is monotonic and positive, for $f(0) = f(\pi) = 0$, we must have $C_1 = C_2 = 0$, so $f \equiv 0$, which is not what we want.

- When $\lambda = 0$, we know that $f''(x) = 0$, so $f(x) = Ax + B$, and for linear equations, for it to be zero twice, it is always zero.

- When $\lambda < 0$, we would have $r = \pm \mathrm{i}\sqrt{-\lambda}$, so the solutions are:

$$f(x) = C_1 \cos(\sqrt{-\lambda}t) + C_2 \sin(\sqrt{-\lambda}t).$$

  By having $f(0) = 0$, we must have $C_1 = 0$, and hence we have $C_2 \sin(\sqrt{-\lambda}\pi) = 0$, hence we know that as long as $\lambda < 0$ and $\sqrt{-\lambda} \in \mathbb{Z}$, this solution is nontrivial.

Hence, for any $\boxed{\lambda < 0 \text{ and } \sqrt{-\lambda}}$, the solutions are $\boxed{C \sin(\sqrt{-\lambda}t)}$.

9. Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(x)y = x^4, \\ y(0) = 0,\ \dfrac{dy}{dx}(0) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 0$.
Hint: Use Existence and Uniqueness Theorem.

Solution.
Here, it is not hard to observe that both $\cos(x)$ and x^4 are continuous over $\mathbb{R}$, so by the Existence and Uniqueness theorem, there exists a unique solution.
Now, let's say $u(x)$ is a solution, that means:

$$u''(x) + \cos x u(x) = x^4, u(0) = u'(0) = 0.$$

We then want to show that $v(x) = u(-x)$ is also a solution.
First, it is not hard to check that:

$$v(0) = u(-0) = 0 \qquad \text{and} \qquad v'(0) = -u'(-0) = 0.$$

Then, when we take the derivative, we have:

$$v''(x) = (u(-x))'' = \big(- u'(-x)\big)' u''(-x).$$

Hence, we can plus v into the differential equation to obtain that:

$$v''(x) + \cos(x)v(x) = u''(-x) + \cos(x)u(-x) = u''(-x) + \cos(-x)u(-x) = (-x)^4 = x^4.$$

Hence, we have shown that $v(x)$ is also a solution of the initial value problem. However, by uniqueness, we must have $u(x) = v(x) = u(-x)$, so u is symmetric about $x = 0$.